Overconfidence and Speculative Bubbles

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Motivated by the behavior of asset prices, trading volume, and price volatility during episodes of asset price bubbles, we present a continuous-time equilibrium model in which overconfidence generates disagreements among agents regarding asset fundamentals. With short-sale constraints, an asset buyer acquires an option to sell the asset to other agents when those agents have more optimistic beliefs. As in a paper by Harrison and Kreps, agents pay prices that exceed their own valuation of future dividends because they believe that in the future they will find a buyer willing to pay even more. This causes a significant bubble component in asset prices even when small differences of beliefs are sufficient to generate a trade. In equilibrium, bubbles are accompanied by large trading volume and high price volatility. Our analysis shows that while Tobin's tax can substantially reduce speculative trading when transaction costs are small, it has only a limited impact on the size of the bubble or on price volatility.

I. Introduction

The behavior of market prices and trading volumes of assets during historical episodes of price bubbles presents a challenge to asset pricing theories. A common feature of these episodes, including the recent

Internet stock boom, is the coexistence of high prices and high trading volume.[1] In addition, high price volatility is frequently observed.[2]

In this paper, we propose a model of asset trading, based on heterogeneous beliefs generated by agents' overconfidence, with equilibria that broadly fit these observations. We also provide explicit links between parameter values in the model, such as trading cost and information, and the behavior of equilibrium prices and trading volume. More generally, our model provides a flexible framework to study speculative trading that can be used to analyze links between asset prices, trading volume, and price volatility.

In the model, the ownership of a share of stock provides an opportunity (option) to profit from other investors' overvaluation. For this option to have value, it is necessary that some restrictions apply to short selling. In reality, these restrictions arise from many distinct sources. First, in many markets, short selling requires borrowing a security, and this mechanism is costly.[3] In particular, the default risk if the asset price goes up is priced by lenders of the security. Second, the risk associated with short selling may deter risk-averse investors. Third, limitations to the availability of capital to potential arbitrageurs may also limit short selling.[4] For technical reasons, we do not deal with short-sale costs or risk aversion. Instead we rule out short sales, although our qualitative results would survive the presence of costly short sales as long as the asset owners can expect to make a profit when others have higher valuations.

Our model follows the basic insight of Harrison and Kreps (1978) that when agents agree to disagree and short selling is not possible, asset prices may exceed their fundamental value. In their model, agents disagree about the probability distributions of dividend streams—the reason for the disagreement is not made explicit. We study overconfi-

dence, the belief of an agent that his information is more accurate than it is, as a source of disagreement. Although overconfidence is only one of the many ways by which disagreement among investors may arise,[5] it is suggested by some experimental studies of human behavior and generates a mathematical framework that is relatively simple. Our model has an explicit solution, which allows us to derive several comparative statics results and restrictions on the dynamics of observables. The model may also be regarded as a fully worked out example of the Harrison-Kreps framework in continuous time, where computations and comparison of solutions are particularly tractable.

We study a market for a single risky asset with limited supply and many risk-neutral agents in a continuous-time model with an infinite horizon. The current dividend of the asset is a noisy observation of a fundamental variable that will determine future dividends. In addition to the dividends, there are two other sets of signals available at each instant. The information is available to all agents; however, agents are divided into two groups, and they differ in the interpretation of the signals. Each group overestimates the informativeness of a different signal and, as a consequence, has distinct forecasts of future dividends.[6] Agents in our model know that their forecasts differ from the forecasts of agents in the other group, but behavioral limitations lead them to agree to disagree. As information flows, the forecasts by agents of the two groups oscillate, and the group of agents that is at one instant relatively more optimistic may become at a future date less optimistic than the agents in the other group. These fluctuations in relative beliefs generate trade.

When evaluating an asset, agents consider their own view of fundamentals and the fact that the owner of the asset has an option to sell the asset in the future to agents in the other group. This option can be exercised at any time by the current owner, and the new owner gets, in turn, another option to sell the asset. These characteristics make the option "American" and give it a recursive structure. The value of the option is the value function of an optimal stopping problem. Since the buyer's willingness to pay is a function of the value of the option that he acquires, the payoff from stopping is, in turn, related to the value of the option. This gives us a fixed-point problem that the option value must satisfy. This difference between the current owner's demand price and his fundamental valuation, which is exactly the resale option value,

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[5] For example, Morris (1996) assumes noncommon priors, and Biais and Bossaerts (1998) examine the role of higher-order beliefs.

[6] "We all have the same information, and we're just making different conclusions about what the future will hold" (Henry Blodget, the former star analyst at Merrill Lynch, quoted in Lewis [2002]).

can be reasonably called a bubble.[7] Fluctuations in the value of the bubble contribute an extra component to price volatility. We emphasize that the bubble in our model is a consequence of the divergence of opinions generated by the overestimation of informativeness of the distinct signals. On average, our agents are neither optimists nor pessimists.

In equilibrium, an asset owner will sell the asset to agents in the other group whenever his view of the fundamental is surpassed by the view of agents in the other group by a critical amount. Passages through this critical point determine turnover. When there are no trading costs, we show that the critical point is zero: it is optimal to sell the asset immediately after the valuation of fundamentals is "crossed" by the valuation of agents in the other group. Our agents' beliefs satisfy simple stochastic differential equations, and it is a consequence of properties of Brownian motion that once the beliefs of agents cross, they will cross infinitely many times in any finite period of time right afterward. This results in a trading frenzy, in which the unconditional average volume in any time interval is infinite. Since the equilibria display continuity with respect to the trading cost c, our model with small trading costs is able to capture the excessive trading observed in bubbles.

When trading costs are small, the value of the bubble and the extra volatility component are maximized. We show that increases in some parameter values, such as the degree of overconfidence or the information content of the signals, increase these three key variables. In this way, our model provides an explanation for the cross-sectional correlation between price, volume, and volatility that has been observed in bubbles. However, to obtain more realistic time-series dynamics, it may be necessary to allow the parameter values to change over time. In Section VII*E*, we discuss how to accommodate fluctuations in parameter values, though we do not provide a theory that explains these oscillations.

In the model, increases in trading costs reduce the trading frequency, asset price volatility, and the option value. For small trading costs, the effect on trading frequency is very significant. The impact on price volatility and on the size of the bubble is much more modest. As the trading cost increases, the increase in the critical point also raises the profit of the asset owner from each trade, thus partially offsetting the decrease in the value of the resale option caused by the reduction in trading frequency. Our analysis suggests that a transaction tax, as proposed by Tobin (1978), would, in fact, substantially reduce the amount of speculative trading in markets with small transaction costs but would

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[7] An alternative would be to measure the bubble as the difference between the asset price and the fundamental valuation of the dividends by an agent that correctly weights the signals. We opted for our definition because it highlights the difference between beliefs about fundamentals and trading price.

have a limited effect on the size of the bubble or on price volatility. Since a Tobin tax will no doubt also deter trading generated by fundamental causes that are absent from our model, the limited impact of the tax on the size of the bubble and on price volatility cannot serve as an endorsement of the Tobin tax. The limited effect of transaction costs on the size of the bubble is also compatible with the observation of Shiller (2000) that bubbles have occurred in the real estate market, where transaction costs are high.[8]

The existence of the option component in the asset price creates potential violations to the law of one price. Through a simple example, we illustrate that the bubble may cause the price of a subsidiary to be larger than that of its parent firm. The intuition behind the example is that if a firm has two subsidiaries with fundamentals that are perfectly negatively correlated, there will be no differences in opinion, and hence no option component on the value of the parent firm, but possibly strong differences of opinion about the value of a subsidiary. In this example, our model also predicts that trading volume on the subsidiaries would be much larger than on the parent firm. This nonlinearity of the option value may help explain the "mispricing" of carve-outs that occurred in the late 1990s such as the 3Com-Palm case.

The paper is structured as follows. In Section II, we present a brief literature review. Section III describes the structure of the model. Section IV derives the evolution of agents' beliefs. In Section V, we discuss the optimal stopping time problem and derive the equation for equilibrium option values. In Section VI, we solve for the equilibrium. Section VII discusses several properties of the equilibrium when trading costs are small. In Section VIII, we focus on the effect of trading costs. In Section IX, we construct an example in which the price of a subsidiary is larger than that of its parent firm. Section X concludes the paper with some discussion of implications for corporate finance. All proofs are in the Appendix.

II. Related Literature

There is a large literature on the effects of heterogeneous beliefs. In a static framework, Miller (1977) and Chen, Hong, and Stein (2002) analyze the overvaluation generated by heterogeneous beliefs. This static framework cannot generate an option value or the dynamics of trading.

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[8] In contrast, Federal Reserve Chairman Alan Greenspan seems to believe that the low turnover induced by the high costs of transactions in the housing market are an impediment to real estate bubbles: "While stock market turnover is more than 100 per cent annually, the turnover of home ownership is less than 10 per cent annually—scarcely tinder for speculative conflagration" (quoted in *Financial Times* [April 22, 2002]). The results in this paper suggest otherwise.

Harris and Raviv (1993), Kandel and Pearson (1995), and Kyle and Lin (2002) study models in which trading is generated by heterogeneous beliefs. However, in all these models there is no speculative component in prices.

Psychology studies suggest that people overestimate the precision of their knowledge in some circumstances, especially for challenging judgment tasks (see Alpert and Raiffa 1982; Lichtenstein, Fischhoff, and Phillips 1982). Camerer (1995) argues that even experts can display overconfidence. A similar phenomenon is *illusion of knowledge*, the fact that persons who do not agree become more polarized when given arguments that serve both sides (Lord, Ross, and Lepper 1979) (see Barber and Odean [2001] and Hirshleifer [2001] for reviews of this literature). In finance, researchers have developed theoretical models to analyze the implications of overconfidence on financial markets (see, e.g., Kyle and Wang 1997; Daniel, Hirshleifer, and Subrahmanyam 1998; Odean 1998; Bernardo and Welch 2001). In these papers, overconfidence is typically modeled as an overestimation of the precision of one's information. We follow a similar approach but emphasize the speculative motive generated by overconfidence.

The bubble in our model, based on the recursive expectations of traders to take advantage of mistakes by others, is quite different from "rational bubbles" (see Blanchard and Watson 1982; Santos and Woodford 1997). In contrast to our setup, rational bubble models are incapable of connecting bubbles with turnover. In addition, in these models, assets must have (potentially) infinite maturity to generate bubbles. Although, for mathematical simplicity, we treat the infinite horizon case, the bubble in our model does not require infinite maturity. If an asset has a finite maturity, the bubble will tend to diminish as maturity approaches, but it would nonetheless exist in equilibrium.

Other mechanisms have been proposed to generate asset price bubbles (e.g., Allen and Gorton 1993; Allen, Morris, and Postlewaite 1993; Horst 2001; Duffie et al. 2002). None of these models emphasize the joint properties of bubble and trading volume observed in historical episodes.

III. The Model

There exists a single risky asset with a dividend process that is the sum of two components. The first component is a fundamental variable that determines future dividends. The second is "noise." The cumulative dividend process D_t satisfies

$$dD_t = f_t dt + \sigma_D dZ_t^D, \tag{1}$$

where Z^D is a standard Brownian motion and σ_D is a constant volatility

parameter. The fundamental variable f is not observable. However, it satisfies

$$df_t = -\lambda(f_t - \bar{f})dt + \sigma_f dZ_t^f, \tag{2}$$

where $\lambda \geq 0$ is the mean reversion parameter, $\bar{f}$ is the long-run mean of f, $\sigma_f > 0$ is a constant volatility parameter, and Z^f is a standard Brownian motion. The asset is in finite supply, and we normalize the total supply to unity.

There are two sets of risk-neutral agents. The assumption of risk neutrality not only simplifies many calculations but also serves to highlight the role of information in the model. Since our agents are risk-neutral, the dividend noise in equation (1) has no direct impact in the valuation of the asset. However, the presence of dividend noise makes it impossible to infer f perfectly from observations of the cumulative dividend process. Agents use the observations of D and any other signals that are correlated with f to infer current f and to value the asset. In addition to the cumulative dividend process, all agents observe a vector of signals s^A and s^B that satisfy

$$ds_t^A = f_t dt + \sigma_s dZ_t^A \tag{3}$$

and

$$ds_t^B = f_t dt + \sigma_s dZ_t^B, \tag{4}$$

where Z^A and Z^B are standard Brownian motions, and $\sigma_s > 0$ is the common volatility of the signals. We assume that all four processes Z^D, Z^f, Z^A, and Z^B are mutually independent.

Agents in group A (B) think of s^A (s^B) as their own signal, although they can also observe s^B (s^A). Heterogeneous beliefs arise because each agent believes that the informativeness of his own signal is larger than its true informativeness. Agents of group A (B) believe that innovations dZ^A (dZ^B) in the signal s^A (s^B) are correlated with the innovations dZ^f in the fundamental process, with ϕ ($0 < \phi < 1$) as the correlation parameter. Specifically, agents in group A believe that the process for s^A is

$$ds_t^A = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^A.$$

Although agents in group A perceive the correct unconditional volatility of the signal s^A, the correlation that they attribute to innovations causes them to overreact to signal s^A. Similarly, agents in group B believe that the process for s^B is

$$ds_t^B = f_t dt + \sigma_s \phi dZ_t^f + \sigma_s \sqrt{1 - \phi^2} dZ_t^B.$$

On the other hand, agents in group A (B) believe (correctly) that in-

novations to s^B (s^A) are uncorrelated with innovations to Z^B (Z^A). We assume that the joint dynamics of the processes D, f, s^A, and s^B in the mind of agents of each group are public information.

Lemma 1 below shows that a larger ϕ increases the precision that agents attribute to their own forecast of the current level of fundamentals. For this reason, we shall refer to ϕ as the overconfidence parameter.[9]

Each group is large, and there is no short selling of the risky asset. To value future cash flows, we may assume either that every agent can borrow and lend at the same rate of interest r or, equivalently, that agents discount all future payoffs using rate r and that each group has infinite total wealth. These assumptions will facilitate the calculation of equilibrium prices.

IV.   Evolution of Beliefs

The model described in the previous section implies a particularly simple structure for the evolution of the difference in beliefs between the groups of traders. We show that this difference is a diffusion with volatility proportional to ϕ.

Since all variables are Gaussian, the filtering problem of the agents is standard. With Gaussian initial conditions, the conditional beliefs of agents in group $C \in \{A, B\}$ are Gaussian with mean $\hat{f}^C$ and variance γ^C. We shall characterize the stationary solution. Standard arguments (e.g., Rogers and Williams 1987, sec. 6.9; Liptser and Shiryaev 1977, theorem 12.7) allow us to compute the variance of the stationary solution and the evolution of the conditional mean of beliefs. The variance of this stationary solution is the same for both groups of agents and equals

$$\gamma \equiv \frac{\sqrt{[\lambda + (\phi\sigma_f/\sigma_s)]^2 + (1 - \phi^2)[(2\sigma_f^2/\sigma_s^2) + (\sigma_f^2/\sigma_D^2)]} - [\lambda + (\phi\sigma_f/\sigma_s)]}{(1/\sigma_D^2) + (2/\sigma_s^2)}.$$

The following lemma justifies associating the parameter ϕ to "overconfidence."

Lemma 1. The stationary variance γ decreases with ϕ.

In addition, the conditional mean of the beliefs of agents in group *A* satisfies

$$d\hat{f}^A = -\lambda(\hat{f}^A - \bar{f})dt + \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^A - \hat{f}^A dt)$$

$$+ \frac{\gamma}{\sigma_s^2}(ds^B - \hat{f}^A dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^A dt). \tag{5}$$

Since *f* mean-reverts, the conditional beliefs also mean-revert. The other three terms represent the effects of "surprises." These surprises can be represented as standard mutually independent Brownian motions for agents in group *A*:

$$dW_A^A = \frac{1}{\sigma_s}(ds^A - \hat{f}^A dt), \tag{6}$$

$$dW_B^A = \frac{1}{\sigma_s}(ds^B - \hat{f}^A dt), \tag{7}$$

and

$$dW_D^A = \frac{1}{\sigma_D}(dD - \hat{f}^A dt). \tag{8}$$

Note that these processes are only Wiener processes in the mind of group *A* agents. Because of overconfidence ($\phi > 0$), agents in group *A* overreact to surprises in s^A.

Similarly, the conditional mean of the beliefs of agents in group *B* satisfies

$$d\hat{f}^B = -\lambda(\hat{f}^B - \bar{f})dt + \frac{\gamma}{\sigma_s^2}(ds^A - \hat{f}^B dt)$$

$$+ \frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s^2}(ds^B - \hat{f}^B dt) + \frac{\gamma}{\sigma_D^2}(dD - \hat{f}^B dt), \tag{9}$$

and the surprise terms can be represented as mutually independent Wiener processes:

$$dW_A^B = \frac{1}{\sigma_s}(ds^A - \hat{f}^B dt),$$

$$dW_B^B = \frac{1}{\sigma_s}(ds^B - \hat{f}^B dt),$$

$$dW_D^B = \frac{1}{\sigma_D}(dD - \hat{f}^B dt).$$

These processes are a standard three-dimensional Brownian motion only for agents in group B.

Since the beliefs of all agents have constant variance, we shall refer to the conditional mean of the beliefs as their *beliefs*. We let g^A and g^B denote the differences in beliefs:

$$g^A = \hat{f}^B - \hat{f}^A, \quad g^B = \hat{f}^A - \hat{f}^B.$$

The next proposition describes the evolution of these differences in beliefs.

PROPOSITION 1.

$$dg^A = -\rho g^A dt + \sigma_g dW_g^A,$$

where

$$\rho = \sqrt{\left(\lambda + \phi\frac{\sigma_f}{\sigma_s}\right)^2 + (1 - \phi^2)\sigma_f^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)},$$

$$\sigma_g = \sqrt{2}\phi\sigma_f,$$

and W_g^A is a standard Wiener process for agents in group A, with innovations that are orthogonal to the innovations of $\hat{f}^A$.

Proposition 1 implies that the difference in beliefs g^A follows a simple mean-reverting diffusion process in the mind of group A agents. In particular, the volatility of the difference in beliefs is zero in the absence of overconfidence. A larger ϕ leads to greater volatility. In addition, $-\rho/2\sigma_g^2$ measures the pull toward the origin. A simple calculation shows that this *mean reversion*[10] decreases with ϕ. A higher ϕ causes an increase in fluctuations of opinions and a slower mean reversion.

In an analogous fashion, for agents in group B, g^B satisfies

$$dg^B = -\rho g^B dt + \sigma_g dW_g^B,$$

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[10] Conley et al. (1997) argue that this is the correct measure of mean reversion.

where W_g^B is a standard Wiener process, and it is independent of innovations to $\hat{f}^B$.

V.   Trading

Fluctuations in the difference of beliefs across agents will induce trading. It is natural to expect that investors that are more optimistic about the prospects of future dividends will bid up the price of the asset and eventually hold the total (finite) supply. We shall allow for costs of trading: a seller pays $c \geq 0$ per unit of the asset sold. This cost may represent an actual cost of transaction or a tax.

At each t, agents in group $C \in \{A, B\}$ are willing to pay p_t^C for a unit of the asset. The presence of the short-sale constraint, a finite supply of the asset, and an infinite number of prospective buyers guarantee that any successful bidder will pay his reservation price.[11] The amount that an agent is willing to pay reflects the agent's fundamental valuation and the fact that he may be able to sell his holdings for a profit at a later date at the demand price of agents in the other group. If we let $o \in \{A, B\}$ denote the group of the current owner, $\bar{o}$ be the other group, and E_t^o be the expectation of members of group o, conditional on the information they have at t, then

$$p_t^o = \sup_{\tau \geq 0} \mathrm{E}_t^o \left[\int_t^{t+\tau} e^{-r(s-t)} dD_s + e^{-r\tau}(p_{t+\tau}^{\bar{o}} - c) \right], \tag{10}$$

where τ is a stopping time, and $p_{t+\tau}^{\bar{o}}$ is the reservation value of the buyer at the time of transaction $t + \tau$.

Since $dD = \hat{f}_t^o dt + \sigma_D dW_D^o$, using the equations for the evolution of the conditional mean of beliefs (eqq. [5] and [9] above), we have that

$$\int_t^{t+\tau} e^{-r(s-t)} dD_s = \int_t^{t+\tau} e^{-r(s-t)} [\bar{f} + e^{-\lambda(s-t)}(\hat{f}_t^o - \bar{f})] ds + M_{t+\tau},$$

where $\mathrm{E}_t^o M_{t+\tau} = 0$. Hence, we may rewrite equation (10) as

$$p_t^o = \max_{\tau \geq 0} \mathrm{E}_t^o \left\{ \int_t^{t+\tau} e^{-r(s-t)} [\bar{f} + e^{-\lambda(s-t)}(\hat{f}_t^o - \bar{f})] ds + e^{-r\tau}(p_{t+\tau}^{\bar{o}} - c) \right\}. \tag{11}$$

We shall start by postulating a particular form for the equilibrium

[11] This observation simplifies our calculations but is not crucial for what follows. We could partially relax the short-sale constraints or the division of gains from trade, provided that it is still true that the asset owner expects to make speculative profits from other investors.

price function, equation (12) below. Proceeding in a heuristic fashion, we derive properties that our candidate equilibrium price function should satisfy. We then construct a function that satisfies these properties and verify that we have produced an equilibrium.[12]

Since all the relevant stochastic processes are Markovian and time-homogeneous and traders are risk-neutral, it is natural to look for an equilibrium in which the demand price of the current owner satisfies

$$p_t^o = p^o(\hat{f}_t^o,\ g_t^o)\ = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + q(g_t^o), \tag{12}$$

with $q > 0$ and $q' > 0$. This equation states that prices are the sum of two components. The first part, $(\bar{f}/r) + [(\hat{f}_t^o - \bar{f})/(r + \lambda)]$, is the expected present value of future dividends from the viewpoint of the current owner. The second is the value of the resale option, $q(g_t^o)$, which depends on the current difference between the beliefs of the other group's agents and the beliefs of the current owner. We call the first quantity the owner's fundamental valuation and the second the value of the resale option. Using (12) in equation (11) and collecting terms, we obtain

$$p_t^o = p^o(\hat{f}_t^o,\ g_t^o)\ = \frac{\bar{f}}{r} + \frac{\hat{f}_t^o - \bar{f}}{r + \lambda} + \sup_{\tau \geq 0} \mathrm{E}_t^o\left\{\left[\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}}) - c\right]e^{-r\tau}\right\}.$$

Equivalently, the resale option value satisfies

$$q(g_t^o)\ = \sup_{\tau \geq 0} \mathrm{E}_t^o\left\{\left[\frac{g_{t+\tau}^o}{r + \lambda} + q(g_{t+\tau}^{\bar{o}}) - c\right]e^{-r\tau}\right\}. \tag{13}$$

Hence to show that an equilibrium of the form (12) exists, it is necessary and sufficient to construct an option value function q that satisfies equation (13). This equation is similar to a Bellman equation. The current asset owner chooses an optimal stopping time to exercise his resale option. Upon the exercise of the option, the owner gets the "strike price" $[g_{t+\tau}^o/(r + \lambda)] + q(g_{t+\tau}^{\bar{o}})$, the amount of excess optimism that the buyer has about the asset's fundamental value and the value of the resale option to the buyer, minus the cost c. In contrast to the optimal exercise problem of American options, the strike price in our problem depends on the resale option value function itself.

It is apparent from the analysis in this section that one could, in principle, treat an asset with a finite life. Equations (10) and (11) would

apply with the obvious changes to account for the finite horizon. However, the option value q will now depend on the remaining life of the asset, introducing another dimension to the optimal exercise problem. The infinite horizon problem is stationary, greatly reducing the mathematical difficulty.

VI. Equilibrium

In this section, we derive the equilibrium option value, duration between trades, and contribution of the option value to price volatility.

A. Resale Option Value

The value of the option $q(x)$ should be at least as large as the gains realized from an immediate sale. The region in which the value of the option equals that of an immediate sale is the stopping region. The complement is the continuation region. In the mind of the risk-neutral asset holder, the discounted value of the option $e^{-rt}q(g_t^o)$ should be a martingale in the continuation region and a supermartingale in the stopping region. Using Ito's lemma and the evolution equation for g^o, we can state these conditions as

$$q(x) \geq \frac{x}{r + \lambda} + q(-x) - c \tag{14}$$

and

$$\tfrac{1}{2}\sigma_g^2 q'' - \rho x q' - rq \leq 0, \tag{15}$$

with equality if (14) holds strictly. In addition, the function q should be continuously differentiable (smooth pasting). We shall derive a smooth function q that satisfies equations (14) and (15) and then use these properties and a growth condition on q to show that in fact the function q solves (13).

To construct the function q, we guess that the continuation region will be an interval $(-\infty, k^*)$, with $k^* > 0$. The variable k^* is the minimum amount of difference in opinions that generates a trade. As usual, we begin by examining the second-order ordinary differential equation that q must satisfy, albeit only in the continuation region:

$$\tfrac{1}{2}\sigma_g^2 u'' - \rho x u' - ru = 0. \tag{16}$$

The following proposition helps us construct an "explicit" solution to equation (16).

PROPOSITION 2. Let

$$h(x) = \tag{17}$$

$$\begin{cases} U\left(\dfrac{r}{2\rho}, \dfrac{1}{2}, \dfrac{\rho}{\sigma_g^2} x^2\right) & \text{if } x \le 0 \\[2ex] \dfrac{2\pi}{\Gamma(\frac{1}{2} + (r/2\rho))\Gamma(\frac{1}{2})} M\left(\dfrac{r}{2\rho}, \dfrac{1}{2}, \dfrac{\rho}{\sigma_g^2} x^2\right) - U\left(\dfrac{r}{2\rho}, \dfrac{1}{2}, \dfrac{\rho}{\sigma_g^2} x^2\right) & \text{if } x > 0, \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function, and $M: R^3 \to R$ and $U: R^3 \to R$ are two Kummer functions described in the Appendix. The function $h(x)$ is positive and increasing in $(-\infty, 0)$. In addition, h solves equation (16) with

$$h(0) = \frac{\pi}{\Gamma(\frac{1}{2} + (r/2\rho))\Gamma(\frac{1}{2})}.$$

Any solution $u(x)$ to equation (16) that is strictly positive and increasing in $(-\infty, 0)$ must satisfy $u(x) = \beta_1 h(x)$ with $\beta_1 > 0$.

We shall also need properties of the function h that are summarized in the following lemma.

LEMMA 2. For each $x \in R$, $h(x) > 0$, $h'(x) > 0$, $h''(x) > 0$, $h'''(x) > 0$, $\lim_{x \to -\infty} h(x) = 0$, and $\lim_{x \to -\infty} h'(x) = 0$.

Since q must be positive and increasing in $(-\infty, k^*)$, we know from proposition 2 that

$$q(x) = \begin{cases} \beta_1 h(x) & \text{for } x < k^* \\[2ex] \dfrac{x}{r+\lambda} + \beta_1 h(-x) - c & \text{for } x \ge k^*. \end{cases} \tag{18}$$

Since q is continuous and continuously differentiable at k^*,

$$\beta_1 h(k^*) - \frac{k^*}{r+\lambda} - \beta_1 h(-k^*) + c = 0,$$

$$\beta_1 h'(k^*) + \beta_1 h'(-k^*) - \frac{1}{r+\lambda} = 0.$$

These equations imply that

$$\beta_1 = \frac{1}{[h'(k^*) + h'(-k^*)](r+\lambda)}, \tag{19}$$

and k^* satisfies

$$[k^* - c(r+\lambda)][h'(k^*) + h'(-k^*)] - h(k^*) + h(-k^*) = 0. \tag{20}$$

The next theorem shows that, for each c, there exists a unique pair (k^*, β_1) that solves equations (19) and (20). The smooth pasting conditions are sufficient to determine the function q and the "trading point" k^*.

THEOREM 1. For each trading cost $c \geq 0$, there exists a unique k^* that solves (20). If $c = 0$, then $k^* = 0$. If $c > 0$, $k^* > c(r + \lambda)$.

The next theorem establishes that the function q described by equation (18), with β_1 and k^* given by (19) and (20), solves (13). The proof consists of two parts. First, we show that (14) and (15) hold and that q' is bounded. We then use a standard argument to show that in fact q solves equation (13) (see, e.g., Kobila [1993] or Scheinkman and Zariphopoulou [2001] for similar arguments).

THEOREM 2. The function q constructed above is an equilibrium option value function. The optimal policy consists of exercising immediately if $g^o > k^*$; otherwise wait until the first time in which $g^o \geq k^*$.

It is a consequence of theorem 2 that the process g^o will have values in $(-\infty, k^*)$. The value k^* acts as a barrier, and when g^o reaches k^*, a trade occurs, the owner's group switches, and the process is restarted at $-k^*$. The function $q(g^o)$ is the difference between the current owner's demand price and his fundamental valuation and can be legitimately called a bubble. When a trade occurs, this difference is

$$b \equiv q(-k^*) = \frac{1}{r + \lambda} \frac{h(-k^*)}{h'(k^*) + h'(-k^*)}. \tag{21}$$

Using equation (21), we can write the value of the resale option as

$$q(x) = \begin{cases} \dfrac{b}{h(-k^*)} h(x) & \text{for } x < k^* \\[3mm] \dfrac{x}{r + \lambda} + \dfrac{b}{h(-k^*)} h(-x) - c & \text{for } x \geq k^*. \end{cases} \tag{22}$$

B.   Duration between Trades

We let $w(x, k, r) = E^o[e^{-r\tau(x,k)}|x]$, with $\tau(x, k) = \inf\{s : g^o_{t+s} > k\}$, given $g^o_t = x \leq k$. The term $w(x, k, r)$ is the discount factor applied to cash flows received the first time the difference in beliefs reaches the level of k given that the current difference in beliefs is x. Standard arguments (e.g., Karlin and Taylor 1981, p. 243) show that w is a nonnegative and strictly monotone solution to

$$\tfrac{1}{2}\sigma_g^2 w_{xx} - \rho x w_x = rw, \quad w(k, k, r) = 1.$$

Therefore, proposition 2 implies that

$$w(x, \ k, \ r) \ = \ \frac{h(x)}{h(k)}. \tag{23}$$

Note that the free parameter β_1 does not affect w.

If $c > 0$, trading occurs the first time $t > s$ when $g_t^o = k^*$ given that $g_s^o = -k^*$. The expected duration between trades provides a useful measure of trading frequency. Since w is the moment-generating function of τ,

$$\mathrm{E}[\tau(-k^*, \ k^*)] \ = \ -\frac{\partial w(-k^*, \ k^*, \ r)}{\partial r}\bigg|_{r=0}.$$

When $c = 0$, the expected duration between trades is zero. This is a consequence of Brownian local time, as we discuss below.

C.   An Extra Volatility Component

The option component introduces an extra source of price volatility. Proposition 1 states that the innovations in the asset owner's beliefs $\hat{f}^o$ and the innovations in the difference of beliefs g^o are orthogonal. Therefore, the total price volatility is the sum of the volatility of the fundamental value in the asset owner's mind, $(\bar{f}/r) + [(\hat{f}_t^o - \bar{f})/(r + \lambda)]$, and the volatility of the option component.

PROPOSITION 3. The volatility from the option value component is

$$\eta(x) \ = \ \frac{\sqrt{2}\phi\sigma_f}{r + \lambda} \frac{h'(x)}{h'(k^*) \ + \ h'(-k^*)} \quad \forall x < k^*. \tag{24}$$

Since $h' > 0$, the volatility of the option value is monotone.

The variance of an agent's valuation of the discounted dividends is

$$\frac{1}{(r + \lambda)^2}\left[\left(\frac{\phi\sigma_s\sigma_f + \gamma}{\sigma_s}\right)^2 + \left(\frac{\gamma}{\sigma_s}\right)^2 + \left(\frac{\gamma}{\sigma_D}\right)^2\right] =$$

$$\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)^{-1}(r + \lambda)^{-2}\left\{2\lambda^2 + \frac{2\lambda\phi\sigma_f}{\sigma_s} + \frac{2\sigma_f^2}{\sigma_s^2} + \frac{\sigma_f^2}{\sigma_D^2}\right.$$

$$\left. - 2\lambda\left[\lambda^2 + \frac{2\lambda\phi\sigma_f}{\sigma_s} + (2 - \phi^2)\frac{\sigma_f^2}{\sigma_s^2} + (1 - \phi^2)\frac{\sigma_f^2}{\sigma_D^2}\right]^{1/2}\right\},$$

which increases with ϕ if $\lambda > 0$ and equals $\sigma_f^2/(r + \lambda)^2$ if $\lambda = 0$. Therefore, an increase in overconfidence increases the volatility of the agent's valuation of discounted dividends. In the remainder of the paper, we ignore

this effect, which vanishes when $\lambda = 0$, to focus on the extra volatility component caused by the option value.

VII.   Properties of Equilibria for Small Trading Costs

In this section, we discuss several of the characteristics of the equilibrium for small trading costs, including the volume of trade and the magnitudes of the bubble and of the extra volatility component. We also provide some comparative statics and show how parameter changes comove price, volatility, and turnover.

A.   Trading Volume

It is a property of Brownian motion that if it hits the origin at t, it will hit the origin at an infinite number of times in any nonempty interval $[t, t + \Delta t)$. In our limit case of $c = 0$, this implies an infinite amount of trade in any nonempty interval that contains a single trade. When the cost of trade $c = 0$, in any time interval, turnover is either zero or infinity, and the unconditional average volume in any time interval is infinity.[13] The expected time between trades depends continuously on c, so it is possible to calibrate the model to obtain any average daily volume. However, a serious calibration would require accounting for other sources of trading, such as shocks to liquidity, and should match several moments of volume, volatility, and prices.

B.   Magnitude of the Bubble

When $c = 0$, a trade occurs each time traders' fundamental beliefs "cross." Nonetheless, the bubble at this trading point is strictly positive since

$$b = \frac{1}{2(r + \lambda)} \frac{h(0)}{h'(0)}.$$

Owners do not expect to sell the asset at a price above their own valuation, but the option has a positive value. This result may seem counterintuitive. To clarify it, it is worthwhile to examine the value of the option when trades occur whenever the absolute value of the differences in fundamental valuations equals an $\epsilon > 0$. An asset owner in group A (B) expects to sell the asset when agents in group B (A) have a fundamental valuation that exceeds the fundamental valuation of agents

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[13] The unconditional probability that it is zero depends on the volatility and mean reversion of the process of the difference of opinions and on the length of the interval. As the length of the interval goes to infinity, the probability of no trade goes to zero.

in group A (B) by ϵ, that is, $g^A = \epsilon$ ($g^A = -\epsilon$). If we write b_0 for the value of the option for an agent in group A that buys the asset when $g^A = -\epsilon$ and b_1 for the value of the option for an agent of group B that buys the asset when $g^A = \epsilon$, then

$$b_0 = \left(\frac{\epsilon}{r+\lambda} + b_1\right)\frac{h(-\epsilon)}{h(\epsilon)},$$

where $h(-\epsilon)/h(\epsilon)$ is the discount factor from equation (23). Symmetry requires that $b_0 = b_1$ and hence

$$b_0 = \frac{\epsilon}{r+\lambda}\frac{h(-\epsilon)}{h(\epsilon) - h(-\epsilon)}.$$

As $\epsilon \to 0$,

$$b_0 \to \frac{1}{2(r+\lambda)}\frac{h(0)}{h'(0)} = b.$$

In this illustration, as $\epsilon \to 0$, trading occurs with higher frequency and the waiting time goes to zero. In the limit, traders will trade infinitely often and the small gains in each trade compound to a significant bubble. This situation is similar to the cost from hedging an option using a stop-loss strategy studied in Carr and Jarrow (1990).

It is intuitive that when σ_g becomes larger, there is more difference of beliefs, resulting in a larger bubble. Also, when ρ becomes larger, for a given level of difference in beliefs, the resale option is expected to be exercised quicker, and therefore there is also a larger bubble. In fact we can show that the following lemma is true.

LEMMA 3. If c is small, b increases with σ_g and ρ and decreases with r and θ. For all $x < k^*$, $q(x) = b[h(x)/h(-k^*)]$ increases with σ_g and ρ and decreases with r and θ.

The proof of lemma 3 actually shows that whenever c is small, the effect of a change in a parameter on the barrier is second-order.

Proposition 1 allows us to write σ_g and ρ using the parameters ϕ, λ, σ_f, $i_s = \sigma_f/\sigma_s$, and $i_D = \sigma_f/\sigma_D$, where i_s and i_D measure the information in each of the two signals and the dividend flow, respectively. To simplify calculations, we set $\lambda = 0$. Then

$$\sigma_g = \sqrt{2}\phi\sigma_f,$$
$$\rho = \sqrt{(2 - \phi^2)i_s^2 + (1 - \phi^2)i_D^2}.$$

Differentiating these equations, one can show the following: (1) As σ_f increases, σ_g increases and ρ is unchanged. Therefore, b and $q(x)$, for $x < k^*$, increase. The bubble increases with the volatility of the funda-

mental process. (2) As i_s or i_D increases, σ_g is unchanged and ρ increases since $0 < \phi < 1$. Therefore, b and $q(x)$, for $x < k^*$, increase. The bubble increases with the amount of information in the signals and the dividend flow. (3) As ϕ increases, σ_g increases and ρ decreases. Thus an increase in ϕ has offsetting effects on the size of the bubble. However, numerical exercises indicate that the size of the bubble always increases with ϕ.

C.   *Magnitude of the Extra Volatility Component*

As the difference of opinions x approaches the trading point, the volatility of the option value approaches

$$\frac{\sqrt{2}\phi\sigma_f}{r + \lambda} \frac{h'(k^*)}{h'(k^*) + h'(-k^*)}.$$

We have the following lemma.

LEMMA 4. If c is small, $\eta(k^*)$ decreases with the interest rate r and the degree of mean reversion λ and increases with the overconfidence parameter ϕ and the fundamental volatility σ_f.

This lemma implies that an increase in the volatility of fundamentals has an additional effect on price volatility at trading points, through an increase in the volatility of the option component.

D.   *Price, Volatility, and Turnover*

Our model provides a link between asset prices, price volatility, and share turnover. Since these are endogenous variables, their relationship will typically depend on which exogenous variable is shifted. In this subsection, we illustrate this link using numerical examples with a small trading cost.

Figure 1 shows the effect of changes in ϕ on the equilibrium when there is a small transaction cost on the trading barrier k^*, expected duration between trades, the bubble at the trading point b, and the volatility of the bubble at the trading point, $\eta(k^*)$. The expected duration between trades is measured in years. The terms k^*, $\eta(k^*)$, and b are measured in multiples of the fundamental volatility $\sigma_f/(r + \lambda)$.[14] Recall that, as ϕ increases, the volatility parameter σ_g in the difference of beliefs increases, whereas the mean reversion parameter ρ decreases. As a result, the resale option becomes more valuable to the asset owner, the bubble and the extra volatility component become larger, and the optimal trading barrier becomes higher. The duration between trades is determined

[14] Since the bubble is generated through an option value, it is natural to normalize it by the volatility of the underlying fundamental value, i.e., the price volatility that would prevail if fundamentals were observable.
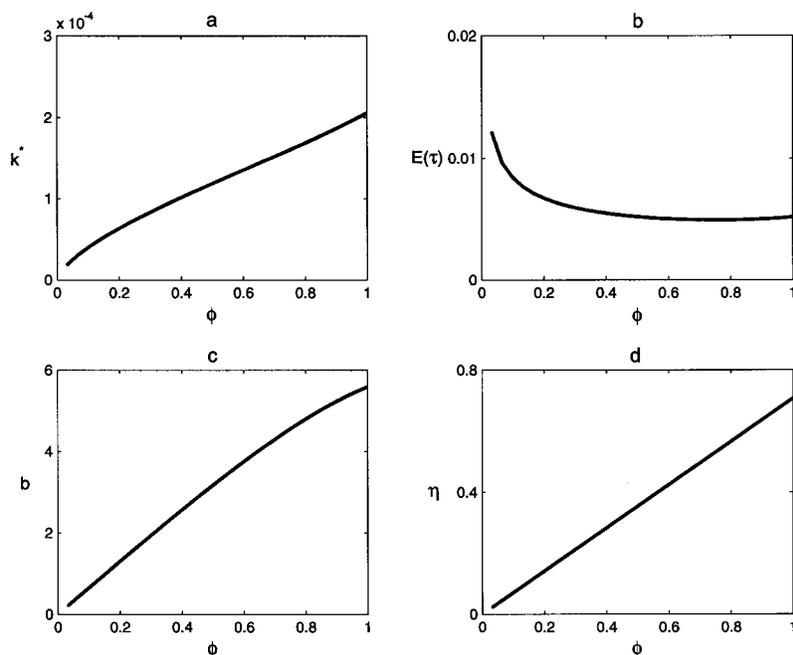
Fig. 1.—Effect of overconfidence level: a, trading barrier; b, duration between trades; c, bubble; d, extra volatility component. Here, $r = 5$ percent, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, $i_D = 0$, and $c = 10^{-6}$. The values of the bubble and the extra volatility component are computed at the trading point. The trading barrier, the bubble, and the extra volatility component are measured as multiples of $\sigma_f/(r + \lambda)$, the fundamental volatility of the asset.

by two offsetting effects as ϕ increases. On the one hand, the trading barrier becomes higher, making the duration between trades longer. On the other hand, the volatility σ_g of the difference in beliefs increases, causing the duration to be shorter. As we stated, the proof of lemma 3 shows that, when c is small, the change in the trading barrier k^* is second-order. Thus the duration between trades typically decreases, as illustrated in figure 1b.

Figure 2 shows the effect of changes in the volatility of the noise in signals σ_s on the equilibrium, again with a small transaction cost. We measure the changes of σ_s in terms of the ratio $i_s = \sigma_f/\sigma_s$. As i_s increases, the mean reversion parameter ρ of the difference in beliefs increases, and the volatility parameter σ_g is unchanged. Intuitively, the increase in ρ causes the trading barrier and the duration between trades to drop. Nevertheless, the bubble at the trading point becomes larger because of the increase in trading frequency. The extra volatility component η is almost independent of i_s since it is essentially determined by ϕ and σ_f as shown in equation (24).
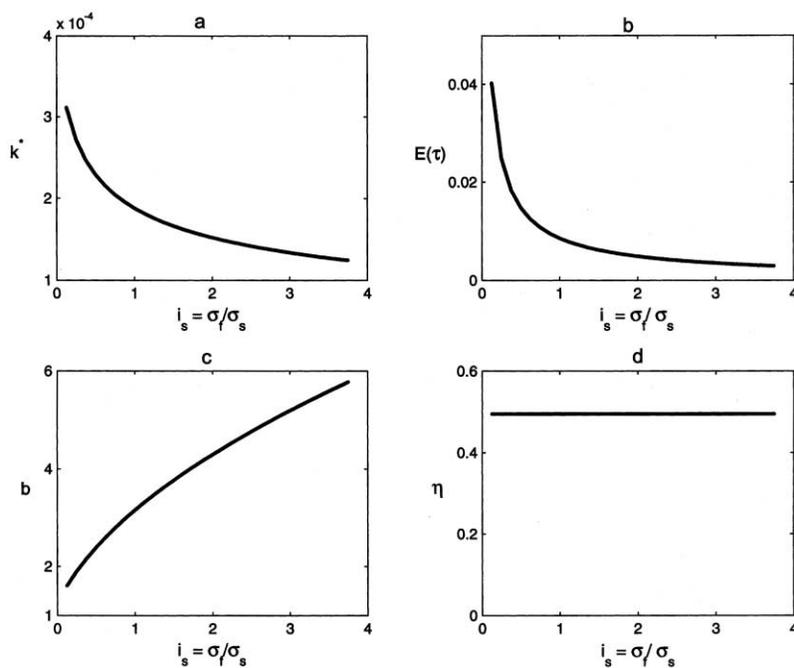
Fig. 2.—Effect of information in signals: *a*, trading barrier; *b*, duration between trades; *c*, bubble; *d*, extra volatility component. Here, $r = 5$ percent, $\lambda = 0$, $\theta = 0.1$, $\phi = 0.7$, $i_D = 0$, and $c = 10^{-6}$. The values of the bubble and the extra volatility component are computed at the trading point. The trading barrier, the bubble, and the extra volatility component are measured as multiples of $\sigma_f/(r+\lambda)$, the fundamental volatility of the asset.

In both cases, there is a monotonically increasing relationship between the size of the bubble at the trading point and duration between trades. In addition, the extra price volatility either increases or does not decrease. We have also verified that this qualitative relationship holds for many other parameter values. In our risk-neutral world, we may consider several assets and analyze the equilibrium in each market independently. In this way our comparative statics properties can be translated into results about correlations among equilibrium variables in the different markets. Thus our model is potentially capable of explaining the observed cross-sectional correlation between log market/book and log turnover for U.S. stocks in the period 1996–2000 as documented by Cochrane (2002) and a similar cross-sectional correlation between the price ratio of China's A shares to B shares and turnover (see Mei, Scheinkman, and Xiong 2003).

E.   Crashes and Fluctuations in Parameters

There are several ways in which we can imagine a change in equilibrium that brings the bubble to zero. The fundamental of the asset may become observable. The overconfident agents may correct their overconfidence. The fundamental volatility of the asset may disappear. For concreteness, imagine that agents in both groups believe that the asset fundamental will become observable at a date determined by a Poisson process that has a parameter θ and is independent of the four Brownian motions described earlier in the model. Once the fundamental becomes observable, agents in each group believe that the beliefs of agents in the other group will collapse to their own. In this case, it is easy to see that the option value

$$q(g_t^o) = \sup_{\tau \geq 0} \mathrm{E}_t^o \left\{ \left[\frac{g_{t+\tau}^o}{r+\lambda} + q(g_{t+\tau}^{\bar{o}}) - c \right] e^{-(r+\theta)\tau} \right\}.$$

Effectively, a higher discount rate $r+\theta$ is used for the profits from exercising the option.

Cochrane (2002) shows that there was a (time-series) correlation between the New York Stock Exchange index and NYSE volume through the 1929 boom and crash and between the NASDAQ index and NASDAQ volume throughout the Internet bubble. To reproduce such a correlation in a nontrivial manner, we would have to generalize our model to account for parameter changes, in the same spirit as our discussion of crashes. For concreteness, imagine that the overconfidence parameter ϕ or the informativeness of signals i_s can assume a finite number of values and that the value of the parameter follows a Markov process with Poisson times that are independent of all the other relevant uncertainty. The model will then produce results that are qualitatively similar to the case in which these parameters are constant, except that the average size of the bubble at any time will depend on the current value of the parameter. In this way, we can admit fluctuations on the size of the bubble and turnover rates, although a more interesting discussion should account for reasons for the parameter fluctuations.

VIII.   Effect of Trading Costs

Using the results established in Section VI*A*, we can show that increasing the trading cost c raises the trading barrier k^* and reduces b, $q(x)$, and $\eta(x)$. In fact, we have the following proposition.

PROPOSITION 4. If c increases, the optimal trading barrier k^* increases. Furthermore, the bubble $q(x)$ and the extra volatility component $\eta(x)$
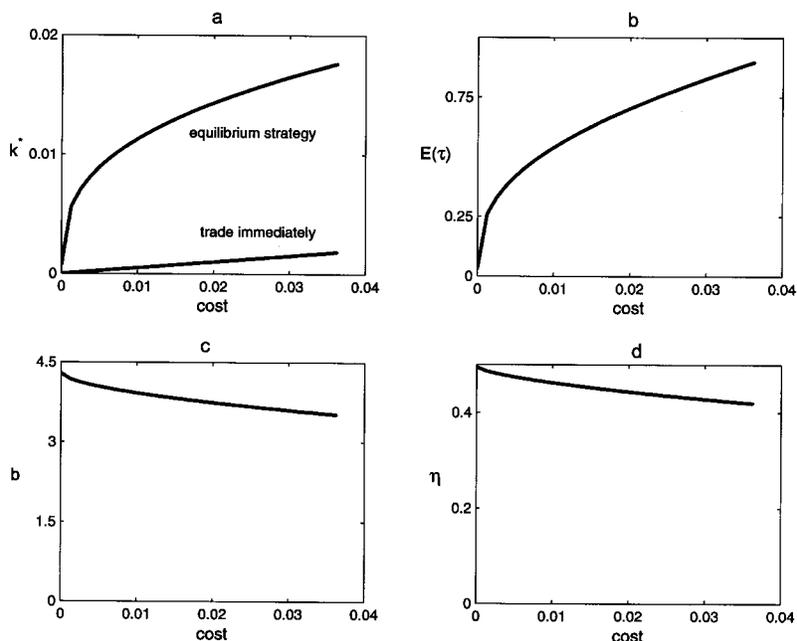
Fig. 3.—Effect of trading costs: a, trading barrier; b, duration between trades; c, bubble; d, extra volatility component. Here, $r = 5$ percent, $\phi = 0.7$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, and $i_D = 0$. The values of the bubble and the extra volatility component are computed at the trading point. The trading barrier, the bubble, the extra volatility component, and trading cost are measured as multiples of $\sigma_f/(r + \lambda)$, the fundamental volatility of the asset.

decrease for all $x < k^*(c)$. As $c \to 0$, $dk^*/dc \to \infty$, but the derivatives of b, $q(x)$, and $\eta(x)$ are always finite.

In order to illustrate the effects of trading costs, we use the following parameter values from our previous numerical exercise: $r = 5$ percent, $\phi = 0.7$, $\lambda = 0$, $\theta = 0.1$, $i_s = 2.0$, and $i_D = 0$. Figure 3 shows the effect of trading costs on the trading barrier k^*, expected duration between trades, the bubble at the trading point b, and the volatility of the bubble at the trading point, $\eta(k^*)$.

Figure 3a shows the equilibrium trading barrier k^*. For comparison, we also graph the amount $c(r + \lambda)$, which corresponds to the difference in beliefs that would justify trade if the option value was ignored. The difference between these two quantities represents the "profits" that the asset owner thinks he is obtaining when he exercises the option to sell. When the trading cost is zero, the asset owner sells the asset immediately when it is profitable, and these profits are infinitely small. As the trading cost increases, the optimal trading barrier increases, and the rate of

increase near $c = 0$ is dramatic since the derivative dk^*/dc is infinite at the origin. As a result, the trading frequency is greatly reduced by the increasing trading cost as shown in figure 3b.

Figures 3c and 3d show that trading costs also reduce the bubble and the extra volatility component, but, as guaranteed by proposition 4, at a limited rate even near $c = 0$. Although one could expect that the strong reduction in trading frequency should greatly reduce the bubble, this effect is partially offset by the increase in profits in each trade.[15] Similar intuition applies to the effect of the trading cost on the extra volatility component.

To estimate the impact of an increase in trading costs, measured as a proportion of price, as opposed to a multiple of fundamental volatility, we must take a stand concerning the relationship between price and volatility of fundamentals. For the parameter values used in our example, figure 3c shows that the bubble at the trading point, for $c \sim 0$, is close to four times the fundamental volatility parameter $\sigma_f/(r + \lambda)$. The drop in prices of Internet stocks from the late 1990s until today exceeds 80 percent. If we take this variation as a measure of the size of the bubble in the late 1990s, then the size of the fundamental volatility also must have been approximately 20 percent of trading prices. In this way, we can reinterpret the values in the figures as multiples of prices. The numerical results indicate that in this case a tax of 1 percent of prices would have caused a reduction of less than 20 percent to the magnitudes of both the bubble and the extra volatility component.

The effectiveness of a trading tax in reducing speculative trading has been hotly debated since Tobin's (1978) initial proposal for a transaction tax in foreign currency markets. Shiller (2000, pp. 225–28) provides an overview of the current status of this debate. Our model implies that for small trading costs, increases in trading costs have a much larger impact in trading frequency than in excess volatility or the magnitude of the price bubble. In reality, trading also occurs for other reasons, such as liquidity shocks or changes in risk-bearing capacity, that are not considered in our analysis; for this reason, the limited impact of transaction costs on volatility and price bubbles cannot serve as an endorsement of a tax on trading. Our numerical exercise can also answer a question raised by Shiller of why bubbles can exist in real estate markets, where the transaction costs are typically high.

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[15] Vayanos (1998) makes a similar point in a different context when analyzing the effects of transaction costs on asset prices in a life cycle model. He shows that an increase in transaction costs can reduce the trading frequency but may even increase asset prices.

IX.   Can the Price of a Subsidiary Be Larger than That of Its Parent Firm?

The existence of the option value component in asset prices can potentially create violations to the law of one price and even make the price of a subsidiary exceed that of a parent company. In this section, we use a simple example to illustrate this phenomenon.

There are two firms, indexed by 1 and 2. For simplicity, we assume that the dividend processes of both assets follow the process in equation (1) with the same parameter σ_D, but with independent innovations and with different fundamental variables f_1 and f_2, respectively. The fundamental variables f_1 and f_2 are unobservable, and both follow the mean-reverting process in equation (2) with the same parameters $\lambda = 0$, $\bar{f}$, and σ_f.

There is a third firm, and the dividend flow of firm 3 is exactly the sum of the dividend flows of firms 1 and 2. In this sense, firms 1 and 2 are both subsidiaries of firm 3. In addition, it is known to all participants that $(f_1)_t + (f_2)_t = f_3$, a constant. This implies that innovations to f_1 and f_2 are perfectly negatively correlated. In particular, the price of a share of firm 3 is f_3/r.

However, according to our analysis, a speculative component exists in the prices of the shares of firms 1 and 2. Since $\hat{f}_1^C + f_2^C = f_3$ for $C \in \{A, B\}$, when agents in group A are holding firm 1, agents in group B must be holding firm 2, and the option components in the prices of these two firms are exactly the same.

The numerical exercise in Section VIID shows that the magnitude of the option component can equal four or five times fundamental volatility. If fundamental volatility is large relative to the discounted value of fundamentals, the value of one of the subsidiaries will exceed the value of firm 3, even though all prices are nonnegative.[16] Although highly stylized, this analysis may help clarify the episodes such as 3Com's equity carve-out of Palm and its subsequent spinoff.[17] In early 2000, for a period of over two months, the total market capitalization of 3Com was significantly less than the market value of its holding in Palm, a subsidiary of 3Com. Other examples of this kind are discussed in Schill and Zhou (2001), Mitchell, Pulvino, and Stafford (2002), and Lamont and Thaler (2003). Our model also predicts that trading in the subsidiary would be much higher than trading in the parent company because of the higher fluctuation in beliefs about the value of the subsidiary. In fact, Lamont and Thaler show that the turnover rate of the subsidiaries'

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[16] Duffie et al. (2002) provide another mechanism to explain this phenomenon based on the lending fee that the asset owner can expect to collect.

[17] The missing link is to demonstrate that the divergence of beliefs on the combined entity was smaller than the divergence of beliefs on the Palm spinoff.

stocks was, on average, six times higher than that of the parent firms' stocks.

This example also indicates that the diversification of a firm reduces the bubble component in the firm's stock price because diversification reduces the fundamental uncertainty of the firm, therefore reducing the potential disagreements among investors. This result is consistent with the diversification discount "puzzle"—the fact that the stocks of diversified firms appear to trade at a discount compared to the stocks of undiversified firms (see, e.g., Lang and Stulz 1994; Berger and Ofek 1995).

X.   Conclusion and Further Discussions

In this paper, we provide a simple model to study bubbles and trading volume that result from speculative trading among agents with heterogeneous beliefs. Heterogeneous beliefs arise from the presence of overconfident agents. With a short-sale constraint, an asset owner has an option to sell the asset to other agents with more optimistic beliefs. Agents value this option and consequently pay prices that exceed their own valuation of future dividends because they believe that in the future they will find a buyer willing to pay even more. We solve the optimal exercise problem of an asset owner and derive, in an almost analytic form, many of the equilibrium variables of interest. This allows us to characterize properties of the magnitude of the bubble, trading frequency, and asset price volatility and to show that the model is consistent with the observation that in actual historical bubbles, volatility and turnover are also inordinate. Theoretical results and numerical exercises suggest that a small trading tax may be effective in reducing speculative trading, but it may not be very effective in reducing price volatility or the size of the bubble. Through a simple example, we also illustrate that the bubble can cause the price of a subsidiary to be larger than that of its parent firm, a violation of the law of one price.

It is natural to conjecture that the existence of a speculative component in asset prices has implications for corporate strategies. Firm managers may be able to profit by adopting strategies that boost the speculative component.

The underpricing of a firm's initial public offering (IPO) has been puzzling. Rajan and Servaes (1997) show that higher initial returns on an IPO lead to more analysts and media coverage. Since investors may disagree about the precision of information provided by the media, the increase in this coverage could increase the option component of the stock. Therefore, IPO underpricing could be a strategy used by firm managers to boost the price of their stocks. If this mechanism is operative, underpricing is more likely to occur when managers hold a

larger share of the firm. This agrees with the empirical results in Aggarwal, Krigman, and Womack (2002). According to our model, a bigger underpricing should be associated with a larger trading volume. In fact, Reese (2000) finds that the higher initial IPO returns are associated with larger trading volume for more than three years after issuance. In a similar fashion, our framework may also be useful for understanding returns and volume on name changes (adding "dot-com").

In addition, if prices contain a large nonfundamental component, many standard views in both corporate finance and asset pricing that use stock prices as a measure of fundamental value will be substantially altered. For example, Bolton, Scheinkman, and Xiong (2002) analyze managerial contracts in such a model. The paper shows that the presence of overconfidence on the part of potential stock buyers could induce incumbent shareholders to use short-term stock compensation to motivate managerial behavior that increases short-term prices at the expense of long-term performance. This provides an alternative to the common view that the recent corporate scandals were caused by a lack of adequate board supervision.

Appendix

Proofs

Proof of Lemma 1

Let $\vartheta(\phi) = \lambda + \phi(\sigma_f/\sigma_s)$ and $\iota(\phi) = (1 - \phi^2)[(2\sigma_f^2/\sigma_s^2) + (\sigma_f^2/\sigma_D^2)]$. Then

$$\frac{d\gamma}{d\phi} \sim \frac{1}{2} \frac{2\vartheta(d\vartheta/d\phi) + (d\iota/d\phi)}{\sqrt{(\vartheta^2 + \iota)}} - \frac{d\vartheta}{d\phi}$$

$$= \left(\frac{\vartheta}{\sqrt{\vartheta^2 + \iota}} - 1\right)\frac{d\vartheta}{d\phi} + \frac{1}{2\sqrt{\vartheta^2 + \iota}}\frac{d\iota}{d\phi} \leq 0.$$

Proof of Proposition 1

The process for g^A can be derived from equations (5) and (9):

$$dg^A = d\hat{f}^B - d\hat{f}^A = -\left(\lambda + \frac{2\gamma + \phi\sigma_s\sigma_f}{\sigma_s^2} + \frac{\gamma}{\sigma_D^2}\right)g^A dt + \frac{\phi\sigma_f}{\sigma_s}(ds^B - ds^A).$$

Using the formula for γ, we may write the mean-reversion parameter as

$$\rho = \sqrt{\left(\lambda + \phi\frac{\sigma_f}{\sigma_s}\right)^2 + (1 - \phi^2)\sigma_f^2\left(\frac{2}{\sigma_s^2} + \frac{1}{\sigma_D^2}\right)}.$$

Using equations (6) and (7), we get

$$dg^A = -\rho g^A dt + \frac{\phi\sigma_f}{\sigma_s}(\sigma_s dW_B^A - \sigma_s dW_A^A).$$

The result follows by writing

$$W_g^A = \frac{1}{\sqrt{2}}(W_B^A - W_A^A).$$

It is easy to verify that innovations to W_g^A are orthogonal to innovations to $\hat{f}^A$ in the mind of agents in group A.

Proof of Proposition 2

Let $v(y)$ be a solution to the differential equation

$$yv''(y) + (\tfrac{1}{2} - y)v'(y) - \frac{r}{2\rho}v(y) = 0. \tag{A1}$$

It is straightforward to verify that $u(x) = v((\rho/\sigma_g^2)x^2)$ satisfies equation (16).

The general solution of equation (A1) is (see Abramowitz and Stegun 1964, chap. 13)

$$v(y) = \alpha M\left(\frac{r}{2\rho}, \frac{1}{2}, y\right) + \beta U\left(\frac{r}{2\rho}, \frac{1}{2}, y\right),$$

where $M(\cdot, \cdot, \cdot)$ and $U(\cdot, \cdot, \cdot)$ are Kummer functions defined as

$$M(a, b, y) = 1 + \frac{ay}{b} + \frac{(a)_2 y^2}{(b)_2 2!} + \cdots + \frac{(a)_n y^n}{(b)_n n!} + \cdots,$$

with $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$; and

$$U(a, b, y) = \frac{\pi}{\sin \pi b}\left[\frac{M(a, b, y)}{\Gamma(1 + a - b)\Gamma(b)} - y^{1-b}\frac{M(1 + a - b, 2 - b, y)}{\Gamma(a)\Gamma(2 - b)}\right].$$

Furthermore, $M_y(a, b, y) > 0$, for all $y > 0$, $M(a, b, y) \to +\infty$, and $U(a, b, y) \to 0$ as $y \to +\infty$.

Given a solution u to equation (16), we can construct two solutions v to equation (A1) by using the values of the function for $x < 0$ and for $x > 0$. We shall denote the corresponding linear combinations of M and U by $\alpha_1 M + \beta_1 U$ and $\alpha_2 M + \beta_2 U$. If these combinations are constructed from the same u, their values and first derivatives must have the same limit as $x \to 0$. To guarantee that $u(x)$ is positive and increasing for $x < 0$, α_1 must be zero. Therefore,

$$u(x) = \beta_1 U\left(\frac{r}{2\rho}, \frac{1}{2}, \frac{\rho}{\sigma_g^2}x^2\right) \quad \text{if } x \leq 0.$$

The solution must be continuously differentiable at $x = 0$. From the definition of the two Kummer functions, we can show that

$$x \to 0 - , \quad u(x) \to \frac{\beta_1 \pi}{\Gamma(\frac{1}{2} + (r/2\rho))\Gamma(\frac{1}{2})}, \qquad u'(x) \to \frac{\beta_1 \pi \sqrt{\rho}}{\sigma_g \Gamma(r/2\rho)\Gamma(\frac{3}{2})},$$

$$x \to 0 + , \quad u(x) \to \alpha_2 + \frac{\beta_2 \pi}{\Gamma(\frac{1}{2} + (r/2\rho))\Gamma(\frac{1}{2})}, \quad u'(x) \to -\frac{\beta_2 \pi \sqrt{\rho}}{\sigma_g \Gamma(r/2\rho)\Gamma(\frac{3}{2})}.$$

By matching the values and first-order derivatives of $u(x)$ from the two sides of $x = 0$, we have

$$\beta_2 = -\beta_1, \quad \alpha_2 = \frac{2\beta_1 \pi}{\Gamma(\frac{1}{2} + (r/2\rho))\Gamma(\frac{1}{2})}.$$

The function h is a solution to equation (16) that satisfies

$$h(0) = \frac{\pi}{\Gamma(\frac{1}{2} + (r/2\rho))\Gamma(\frac{1}{2})} > 0,$$

and $h(-\infty) = 0$. Equation (16) guarantees that at any critical point at which $h < 0$, h has a maximum, and at any critical point at which $h > 0$, it has a minimum. Hence h is strictly positive and increasing in $(-\infty, 0)$.

Proof of Lemma 2

Write $\alpha = \sigma_g^2/2\rho > 0$ and $\beta = r/\rho > 0$. The function $h(x)$ is a positive, and increasing in $(-\infty, 0)$, solution to $\alpha h'' - x h' - \beta h = 0$.

If $x^* \in R$ with $h(x^*) > 0$ and $h'(x^*) = 0$, then $h''(x^*) = \beta h(x^*)/\alpha > 0$. Hence h has no local maximum while it is positive, and as a consequence, it is always positive and has no local maxima. In particular, h is monotonically increasing. Since $h' > 0$ for $x \le 0$ and $h'' \ge 0$ for $x \ge 0$, $h'(x) > 0$ for all x. From the solution constructed in proposition 2, $\lim_{x \to -\infty} h(x) = 0$.

Note that any solution to the differential equation is infinitely differentiable. Next, we show that h is convex. For $x > 0$, $h''(x) = [x h'(x)/\alpha] + [\beta h(x)/\alpha] > 0$. To prove that h is also convex for $x < 0$, let us assume that there exists $x^* < 0$ such that $h''(x^*) \le 0$. Then

$$h'''(x^*) = \frac{x^* h''(x^*)}{\alpha} + \frac{(\beta + 1) h'(x^*)}{\alpha} > 0.$$

This directly implies that $h''(x) < 0$ for $x < x^*$. Then $\lim_{x \to -\infty} h'(x) = \infty$. In this situation the boundary condition $h(-\infty) = 0$ cannot be satisfied. In this way, we get a contradiction.

Let $v(x) = h'(x)$. The function $v(x)$ is positive and increasing. Also, v satisfies $\alpha v''(x) - x v'(x) - (\beta + 1) v(x) = 0$. By repeating the proof that we use for h, we can show that $v(x)$ is convex and $\lim_{x \to -\infty} v(x) = 0$. In fact, one can show that any higher-order derivative of $h(x)$ is positive, increasing, and convex.

Proof of Theorem 1

Let

$$l(k) = [k - c(r + \lambda)][h'(k) + h'(-k)] - h(k) + h(-k).$$

We first show that there exists a unique k^* that solves $l(k) = 0$.

If $c = 0$, then $l(0) = 0$ and $l'(k) = k[h''(k) - h''(-k)] > 0$ for all $k \neq 0$. Therefore, $k^* = 0$ is the only root of $l(k) = 0$.

If $c > 0$, then $l(k) < 0$ for all $k \in [0, c(r + \lambda)]$. Since h'' and h''' are increasing (lemma 2), for all $k > c(r + \lambda)$,

$$l'(k) = [k - c(r + \lambda)][h''(k) - h''(-k)] > 0,$$

$$l''(k) = h''(k) - h''(-k) + [k - c(r + \lambda)][h'''(k) - h'''(-k)] > 0.$$

Therefore, $l(k) = 0$ has a unique solution $k^* > c(r + \lambda)$.

Proof of Theorem 2

First we show that q satisfies equation (14). Using the notation introduced in equations (21) and (22), we have

$$q(-x) = \begin{cases} \dfrac{b}{h(-k^*)} h(-x) & \text{for } x > -k^* \\[3mm] \dfrac{-x}{r + \lambda} + \dfrac{b}{h(-k^*)} h(x) - c & \text{for } x \leq -k^*. \end{cases}$$

We must establish that

$$U(x) = q(x) - \frac{x}{r + \lambda} - q(-x) + c \geq 0.$$

A simple calculation shows that

$$U(x) = \begin{cases} 2c & \text{for } x < -k^* \\[3mm] \dfrac{-x}{r + \lambda} + \dfrac{b}{h(-k^*)} [h(x) - h(-x)] + c & \text{for } -k^* \leq x \leq k^* \\[3mm] 0 & \text{for } x > k^*. \end{cases}$$

Thus

$$U''(x) = \frac{b}{h(-k^*)} [h''(x) - h''(-x)], \quad -k^* \leq x \leq k^*.$$

From lemma 2 we know that $U''(x) > 0$ for $0 < x < k^*$ and $U''(x) < 0$ for $-k^* < x < 0$. Since $U'(k^*) = 0$, $U'(x) < 0$ for $0 < x < k^*$. On the other hand, $U'(-k^*) = 0$, so $U'(x) < 0$ for $-k^* < x < 0$. Therefore, $U(x)$ is monotonically decreasing for $-k^* < x < k^*$. Since $U(-k^*) = 2c > 0$ and $U(k^*) = 0$, $U(x) > 0$ for $-k^* < x < k^*$.

We now show that equation (15) holds. By construction, it holds in the region

$x \leq k^*$. Therefore, we need only to show that equation (15) is valid for $x \geq k^*$. In this region,

$$q(x) = \frac{x}{r+\lambda} + \frac{b}{h(-k^*)} h(-x) - c;$$

thus

$$q'(x) = \frac{1}{r+\lambda} - \frac{b}{h(-k^*)} h'(-x)$$

and

$$q''(x) = \frac{b}{h(-k^*)} h''(-x).$$

Hence,

$$\frac{1}{2}\sigma_g^2 q''(x) - \rho x q'(x) - rq(x) = \frac{b}{h(-k^*)} \left[\frac{1}{2}\sigma_g^2 h''(-x) + \rho x h'(-x) - rh(-x)\right]$$

$$- \frac{r+\rho}{r+\lambda} x + rc$$

$$= -\frac{r+\rho}{r+\lambda} x + rc \leq -(r+\rho)c + rc$$

$$= -\rho c < 0,$$

where the inequality comes from the fact that $x \geq k^* > (r+\lambda)c$ from theorem 1.

Also, q has an increasing derivative in $(-\infty, k^*)$ and has a derivative bounded in absolute value by $1/(r+\lambda)$ in (k^*, ∞). Hence q' is bounded.

If τ is any stopping time, the version of Ito's lemma for twice-differentiable functions with absolutely continuous first derivatives (e.g., Revuz and Yor 1999, chap. 6) implies that

$$e^{-r\tau}q(g_\tau^o) = q(g_0^o) + \int_0^\tau [\tfrac{1}{2}\sigma_g^2 q''(g_s^o) - \rho g_s^o q'(g_s^o) - rq(g_s^o)]ds$$

$$+ \int_0^\tau \sigma_g q'(g_s^o)dW_s.$$

Equation (15) states that the first integral is nonpositive, and the bound on q' guarantees that the second integral is a martingale. Using equation (14), we obtain

$$E^o\left\{e^{-r\tau}\left[\frac{g_\tau^o}{r+\lambda} + q(-g_\tau^o) - c\right]\right\} \leq E^o[e^{-r\tau}q(g_\tau^o)] \leq q(g_0^o).$$

This shows that no policy can yield more than $q(x)$.

Now consider the stopping time $\tau = \inf\{t : g_t^o \geq k^*\}$. Such a τ is finite with probability one, and g_s^o is in the continuation region for $s < \tau$. Using exactly the

same reasoning as above, but recalling that in the continuation region (15) holds with equality, we obtain

$$q(g^o) = E^o \left\{ e^{-r\tau} \left[\frac{g^o_\tau}{r + \lambda} + q(-g^o_\tau) - c \right] \right\}.$$

Proof of Proposition 3

Since

$$q(x) = \frac{1}{r + \lambda} \frac{h(x)}{h'(k) + h'(-k)},$$

the volatility of $q(g^o_t)$ is given by

$$\frac{1}{r + \lambda} \frac{h'(g^o_t)}{h'(k) + h'(-k)}$$

multiplied by the volatility of g^o_t. From the proof of proposition 1,

$$dg^o_t = -\rho g^o_t dt + \frac{\phi \sigma_f}{\sigma_s}(ds^{\bar{o}} - ds^o).$$

From equations (3) and (4), the volatility of $s^{\bar{o}} - s^o$ is $\sqrt{2}\sigma_s$ in an objective measure. Hence the volatility of g^o is $\sqrt{2}\phi\sigma_f$.

Proof of Lemma 3

When $c = 0$, the magnitude of the bubble at the trading point is

$$b_0 = \frac{\sigma_g}{2\sqrt{2\rho}(r + \lambda)} \frac{\Gamma((r + \theta)/2\rho)}{\Gamma(\frac{1}{2} + [(r + \theta)/2\rho])}.$$

It is obvious that b_0 increases with σ_g. We can directly show that b_0 increases with ρ and decreases with r and θ by plotting it.

When $c = 0$, the bubble is $q_0(x) = b_0[h(x)/h(0)]$, where $h(x)$ is a positive and increasing solution to

$$\tfrac{1}{2}\sigma_g^2 h''(x) - \rho x h'(x) - (r + \theta)h(x) = 0,$$

$$h(0) = \frac{\pi}{\Gamma(\frac{1}{2} + [(r + \theta)/2\rho])\Gamma(\frac{1}{2})}.$$

Note that $q_0(x)$ is not affected by letting $h(0) = 1$.

Assume $\tilde{\sigma}_g > \sigma_g$, let $\tilde{h}(x)$ solve $\tfrac{1}{2}\tilde{\sigma}_g^2 \tilde{h}''(x) - \rho x \tilde{h}'(x) - (r + \theta)\tilde{h}(x) = 0$, $\tilde{h}(-\infty) = 0$, and

$$\tilde{h}(0) = \frac{\pi}{\Gamma(\frac{1}{2} + [(r + \theta)/2\rho])\Gamma(\frac{1}{2})}.$$

We show that $\tilde{h}(x) > h(x)$ for all $x < 0$. Let $f(x) = \tilde{h}(x) - h(x)$. Then from lemma 2, $f(-\infty) = f(0) = 0$. Suppose that f has a local minimum x^* with $f(x^*) < 0$. If

such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, from the equations satisfied by $\tilde{h}(x)$ and $h(x)$, we have

$$\frac{1}{2}[\tilde{\sigma}_g^2 \tilde{h}''(x) - \sigma_g^2 h''(x)] - \rho x[\tilde{h}'(x) - h'(x)] - (r+\theta)[\tilde{h}(x) - h(x)] = 0.$$

This implies that $\tilde{\sigma}_g^2 \tilde{h}''(x^*) < \sigma_g^2 h''(x^*)$. Since $\tilde{\sigma}_g^2 > \sigma_g^2$, this in turn implies that $\tilde{h}''(x^*) < h''(x^*)$. This is equivalent to $f''(x^*) < 0$, which contradicts with x^* as a local minimum. Therefore, $f(x)$ cannot have a local minimum with its value less than zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x \in (-\infty, 0)$. Therefore, $\tilde{h}(x) > h(x)$ for all $x < 0$. This implies that the bubble $q_0(x)$ increases with σ_g for all $x < 0$.

Assume $\bar{\rho} > \rho$, and let $\bar{h}(x)$ solve

$$\frac{1}{2}\sigma_g^2 \bar{h}''(x) - \bar{\rho} x \bar{h}'(x) - (r+\theta)\bar{h}(x) = 0,$$

$\bar{h}(-\infty) = 0$, and

$$\bar{h}(0) = \frac{\pi}{\Gamma(\frac{1}{2} + [(r+\theta)/2\rho])\Gamma(\frac{1}{2})}.$$

We can show that $\bar{h}(x) < h(x)$ for all $x < 0$. Again let $f(x) = \bar{h}(x) - h(x)$. We first establish that $f(x)$ has no local minimum x^* with $f(x^*) < 0$. If such a local minimum exists, $f'(x^*) = 0$ and $f''(x^*) \geq 0$. On the other hand, taking differences, we obtain

$$\frac{1}{2}\sigma_g^2 [\bar{h}''(x) - h''(x)] - \rho x[\bar{h}'(x) - h'(x)] - (r+\theta)[\bar{h}(x) - h(x)] = (\bar{\rho} - \rho) x \bar{h}'(x).$$

This last equation implies that $\bar{h}'(x^*) < 0$, which contradicts the fact that $\bar{h}(x)$ is an increasing function. Therefore, $f(x)$ cannot have a local minimum below zero. Since $f(-\infty) = f(0) = 0$, $f(x)$ must stay above zero for $x < 0$. This directly implies that $\bar{h}(x) > h(x)$ for all $x < 0$, and $q_0(x)$ increases with ρ for all $x < 0$. Similarly, we can prove that $q_0(x)$ decreases with r and θ for all $x < 0$.

One can extend the comparative statics we established for $c = 0$ for the case of c small. Let $\zeta \in \{\sigma_g, \rho, \theta\}$. From equation (21) it follows that if $\partial k^*(\zeta, c)/\partial\zeta = o(k^*)$, then the comparative statics of b with respect to ζ is preserved for small c.

Using the definition of function h in equation (17), we write h as $h(x, \zeta)$. From equation (20),

$$\frac{\partial k^*(\zeta, c)}{\partial\zeta} =$$

$$\frac{\left[\dfrac{\partial h(k^*, \zeta)}{\partial\zeta} - \dfrac{\partial h(-k^*, \zeta)}{\partial\zeta}\right] - [k^* - c(r+\lambda)]\left[\dfrac{\partial^2 h(k^*, \zeta)}{\partial x\partial\zeta} + \dfrac{\partial^2 h(-k^*, \zeta)}{\partial x\partial\zeta}\right]}{[k^* - c(r+\lambda)]\left[\dfrac{\partial^2 h(k^*, \zeta)}{\partial x^2} - \dfrac{\partial^2 h(-k^*, \zeta)}{\partial x^2}\right]}. \quad (A2)$$

As $c \to 0$, $k^* \to 0$ and the numerator and denominator go to zero. To find the

limit behavior, we use the explicit form of h given in the proof of proposition 2 and write

$$h(x, \zeta) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + o(x^4),$$

with

$$C_0 = \frac{\pi}{\Gamma((r/2\rho) + \frac{1}{2})\Gamma(\frac{1}{2})}, \qquad C_1 = \frac{\pi\sqrt{\rho}}{\Gamma(r/2\rho)\Gamma(\frac{3}{2})\sigma_g},$$

$$C_2 = \frac{\pi r}{4\Gamma((r/2\rho) + \frac{1}{2})\Gamma(\frac{1}{2})\sigma_g^2}, \quad C_3 = \frac{\pi\sqrt{\rho}(r + \rho)}{3\Gamma(r/2\rho)\Gamma(\frac{3}{2})\sigma_g^3}.$$

We use equation (20) to replace the term $k^* - c(r + \lambda)$ on the right-hand side of equation (A2) by

$$\frac{h(k^*, \zeta) - h(-k^*, \zeta)}{[\partial h(k^*, \zeta)/\partial x] + [\partial h(-k^*, \zeta)/\partial x]}.$$

Taking limits as $k^* \to 0$, we obtain

$$\frac{\partial k^*(\zeta, c)}{\partial \zeta} \sim o(k^*), \quad \zeta \in \{\sigma_g, \rho, \theta\}.$$

A small variation establishes the same result for $\partial k^*(r, c)/\partial r$. Hence, for small c, b increases with σ_g and ρ and decreases with r and θ. In addition, we can show that $q(x)$ for $x < k^*$ increases with σ_g and ρ and decreases with r and θ.

Proof of Lemma 4

This is analogous to the proof of lemma 3.

Proof of Proposition 4

Let

$$l(k, c) = [k - c(r + \lambda)][h'(k) + h'(-k)] - h(k) + h(-k).$$

The function $k^*(c)$ is the root of $l(k, c) = 0$. If $c > 0$,

$$\frac{dk^*}{dc} = \frac{r + \lambda}{k^* - c(r + \lambda)} \frac{h'(k^*) + h'(-k^*)}{h''(k^*) - h''(-k^*)} > 0.$$

Hence $k^*(c)$ is differentiable in $(0, \infty)$. Now suppose that $c_n \to 0$. The sequence $k^*(c_n)$ is bounded, and every limit point $\bar{k}^*$ must solve $l(\bar{k}^*, 0) = 0$. Hence, as we argued in the proof of theorem 1, $\bar{k}^* = 0$ and the function $k^*(c)$ is continuous.

Hence $dk^*/dc \to \infty$ as $c \to 0$. The claims on b and $q(x)$ follow from equations (21) and (22) and lemma 2. The derivative of $\eta(x)$ with respect to c is

$$\frac{d\eta(x)}{dc} = \frac{\sqrt{2}\phi\sigma_f h'(x)[h''(k^*) - h''(-k^*)]}{r+\lambda} \left(-\frac{dk^*}{dc}\right)$$

$$= -\frac{\sqrt{2}\phi\sigma_f h'(x)}{[k^* - c(r+\lambda)][h'(k^*) + h'(-k^*)]} < 0.$$

Therefore, $\eta(x)$ decreases with c. However, note that $d\eta(x)/dc$ is finite as $c \to 0$, although $dk^*/dc \to \infty$ as $c \to 0$.

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