Waves in Thin Liquid Films

Jeffrey Santner

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Professor Howard Stone

Introduction

Waves in thin films behave differently than waves in deeper water. This is because of the high viscous stresses, low impact of intertial forces, low velocity, and influence of surface tension at the small scales involved. In this paper, the governing equations of thin films will be derived from the Navier-Stokes equations. Then, the behavior of different types of waves will be investigated.

The Lubrication Approximation

When investigating thin films, the full Navier-Stokes equations (equation (1)) can be simplified to make the system easier to solve.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+\rho\mathbf{g}+\mu\nabla^2\mathbf{u} \qquad (1)$$

The velocity of a thin film is very slow due to high viscous forces caused by the small length scales. The inertial terms on the left side of the equation are negligible, because they are proportional to V^2/L, where L is a characteristic length and V is a characteristic (small) velocity. This is obvious for the second term ($\boldsymbol{u}\cdot\nabla\boldsymbol{u}$), and is true in the first term because a characteristic time is L/V, so $\frac{d\boldsymbol{u}}{dt}\propto V^2/L$. Without these inertial terms, the Navier-Stokes equation is given by $0=-\nabla p+\rho\boldsymbol{g}+\mu\nabla^2\boldsymbol{u}$. At the bottom of the film, where the fluid meets the solid, there is a no-slip condition. However, the film can support a flow, so there is a velocity change between the liquid-solid boundary and the liquid-gas boundary, a very short distance. The short distance in the z direction makes the second derivative with respect to z much larger than derivatives in the x and y directions. So, the equation is now $0=-\nabla p+\rho\boldsymbol{g}+\frac{\partial^2\boldsymbol{u}}{\partial z^2}$. The velocity perpendicular to the plane is negligible, so the equations are reduced to equations (2):

$$
\begin{aligned}
-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial z^2}&=0\\
-\frac{dp}{dy}+\mu\frac{\partial^2 v}{\partial z^2}&=0\\
-\frac{dp}{dz}+\rho g&=0
\end{aligned}
\tag{2}
$$

Conservation of mass must also be used. For a thin film with incompressible flow in the x direction, the increase in volumetric flow in the x direction is $\Delta Q=\left(\int_0^{h(x)} u dz\right)_x^{x+dx}$.

This must equal the decrease in volumetric flow in the z direction, which is $-\frac{\partial h}{\partial t}dx$ (Figure 1).

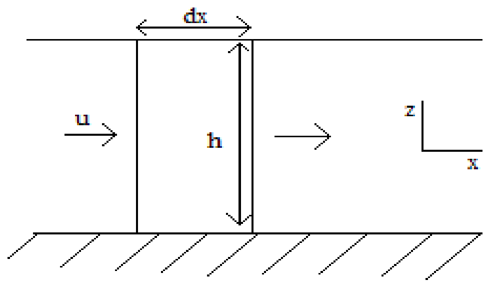


Figure 1: Flow in thin film

Equating the volumetric flows and rearranging gives equation (3).

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\int_0^{h(x)} u dz = -\frac{\partial h}{\partial t} \tag{3}$$

Integrating equation (2) twice, assuming no slip at the wall and no shear at the liquid-gas interface gives the thin film horizontal velocity profile.

$$u = \frac{1}{\mu}\frac{\partial p}{\partial x}\left(\frac{z^2}{2} - hz\right) \tag{4}$$

Applying equation (3) to this velocity profile gives the behavior of the film height.

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right) \tag{5}$$

Decay of Capillary Waves

Now, the behavior of a thin film subject to an initial sinusoidal disturbance will be investigated.

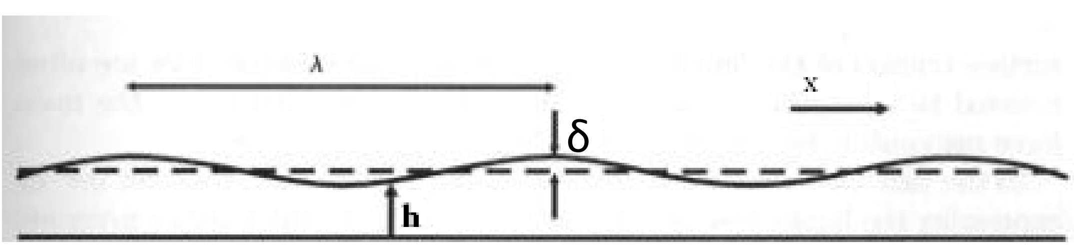


Figure 2: Disturbance on thin film (Gennes et al, 2003)

The height is $h(x,t)=h_0+\delta(t)\cos(qx)$, where q is the wave number (the wavelength is $\lambda=2\pi/q$) and the amplitude is $\delta(t) << h_0$. Assuming that $h << \lambda$, the film is thin and level enough that the lubrication approximation can be used.

The pressure in the fluid is the atmospheric pressure added to gravitational effects and capillary effects. Because the wavelength is large the slope is small, so the curvature can be approximated as $\kappa \approx -d^2h/dx^2$. So, the difference in pressure between a crest and a trough due to capillary effects is of the order $\gamma\delta/\lambda^2$, whereas the pressure difference due to gravity of the order $\rho g\delta$. The ratio of gravitational effects to capillary effects is $\lambda^2\rho g/\gamma$, or $(\lambda/l_{cap})^2$. It should be noted that capillary and gravitational effects both raise the pressure under a crest and decrease it under a trough, working together to push fluid towards the troughs, leveling the film.

Using a length scale in the x direction of λ and a length scale in the z direction of h_0, this equations (2) and (3) can be used to find scaling laws for the average velocity U, mass flow Q, and $d\delta/dt$ as shown below.

Gravity dominates, $(\lambda/l_{cap})^2>>1$	Capillarity dominates, $(\lambda/l_{cap})^2<<1$
$U \propto \frac{h_0^2\rho g\delta}{\lambda\mu}$ $Q \propto \frac{h_0^3\rho g\delta}{\lambda\mu}$ $\frac{d\delta}{dt} \propto -\frac{h_0^3\rho g\delta}{\lambda^2\mu} \propto -\delta\frac{h_0^3q^2\rho g}{\mu}$	$U \propto \frac{h_0^2\gamma\delta}{\lambda^3\mu}$ $Q \propto \frac{h_0^3\gamma\delta}{\lambda^3\mu}$ $\frac{d\delta}{dt} \propto -\frac{h_0^3\gamma\delta}{\lambda^4\mu} \propto -\delta\frac{h_0^3q^4\gamma}{\mu}$

In the above derivation, $\frac{dh}{dt}$ has been replaced with $\frac{d\delta}{dt}$ because they are proportional. Both of these scaling laws are of the form $\frac{d\delta}{dt} \propto -\frac{\delta}{\tau}$, where τ is the time constant of the decaying exponential solution $\delta = \delta_0 e^{-t/\tau}$. So, in the gravitationally and capillary driven cases, the amplitude of the ripple decays exponentially with a time constant according to equation (6).

$$\tau \propto \begin{cases} \dfrac{\mu}{h_0^3 q^2 \rho g} \ , \ \left(\lambda / l_{cap}\right)^2 >> 1 \\ \dfrac{\mu}{h_0^3 q^4 \gamma} \ , \left(\lambda / l_{cap}\right)^2 << 1 \end{cases} \qquad (6)$$

In both cases, the time constant is a strong inverse function of the thickness, and a weak function of viscosity. The wavelength of the disturbance has a much greater impact on the time constant in the capillary driven case than the gravitational case. This is because the capillary force depends on curvature, which depends on q^2 while the gravitational force is not affected by the wave number. In both cases, higher wave numbers correspond to faster decay. An arbitrary disturbance can be represented as a Fourier series, so an arbitrary disturbance will have its high wave number portions decay faster than its low wave number components, smoothing the film towards a pure sine wave until the flat film is reached.

Now, the behavior of δ will be derived more exactly, to check the scaling laws and find the constants of proportionality. The pressure in the fluid can be expressed as

$p - p_{atm} = \rho g h - \gamma \frac{d^2 h}{dx^2} + f(z)$ when using the small slope approximation for curvature.

Substituting this into equation (5) gives equation (7).

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu}\frac{\partial}{\partial x}\left(h^3 \frac{\partial}{\partial x}\left(\rho g h - \gamma \frac{d^2 h}{dx^2}\right)\right)$$

$$\frac{\partial h}{\partial t} = \frac{1}{3\mu}\left(3h^2 \frac{\partial h}{\partial x}\left(\rho g \frac{\partial h}{\partial x} - \gamma \frac{d^3 h}{dx^3}\right) + h^3\left(\rho g \frac{\partial^2 h}{\partial x^2} - \gamma \frac{d^4 h}{dx^4}\right)\right) \tag{7}$$

Substituting h(x,t)=h_0+δ(t)cos(qx), and replacing powers of h with powers of h_0 gives

$$\frac{\partial \delta}{\partial t}\cos qx = \frac{1}{3\mu}\left(-3{h_0}^2 \delta q \sin qx\left(-\rho g \delta q \sin qx - \gamma q^3 \delta \sin qx\right) + {h_0}^3\left(-\rho g \delta q^2 \cos qx - \gamma \delta q^4 \cos qx\right)\right)$$

$$\frac{\partial \delta}{\partial t}\cos qx = \frac{\delta {h_0}^2 q^2}{3\mu}\left(\rho g + \gamma q^2\right)\left(3\delta \sin^2 qx - h_0 \cos qx\right)$$

Because $\delta \ll h_0$, the second to last term can be neglected. Rearranging terms gives equation (8). This result agrees with equation (6) in the gravitationally or capillary driven limits.

$$\frac{\partial \delta}{\partial t} = -\delta \frac{{h_0}^3 q^4 \rho g}{3\mu}\left(\frac{1}{q^2} + {l_{cap}}^2\right) \tag{8}$$

Propagation of Waves

In this section, the behavior of a small moving disturbance will be derived. This topic has been covered in detail using cylindrical polar coordinates in an axially symmetric situation, taking into account surfactants and surface slip in by Tsekov et al (1998). It has also been covered in a very general form, and in many different cases by

Oron et al (1997). In this examination, two dimensional cartesian coordinates will be used, and the no-slip condition will be assumed.

The system under investigation is a thin liquid film, with a disturbance such that $h(x,t)=h_0+\varepsilon(x,t)$, where h is the film thickness, h_0 is the average film thickness, and ε is a small disturbance such that $\varepsilon^2<<h_0^2$. Note that the system is independent of the y direction. Because the disturbance is small and the film is thin, the lubrication approximation can be used. The pressure is the same as the pressure expression used for the relaxation of a disturbance, so equation (7) may be used again, giving the following.

$$\frac{\partial\varepsilon}{\partial t}=\frac{\gamma}{3\mu}\left(3h_0^{\,2}\frac{\partial\varepsilon}{\partial x}\left(\frac{\rho g}{\gamma}\frac{\partial\varepsilon}{\partial x}-\frac{d^3\varepsilon}{dx^3}\right)+h_0^{\,3}\left(\frac{\rho g}{\gamma}\frac{\partial^2\varepsilon}{\partial x^2}-\frac{d^4\varepsilon}{dx^4}\right)\right)$$

To make this equation dimensionless, lengths will be scaled to the capillary length, and time will be scaled to the capillary length divided by the characteristic velocity, $\frac{l_{cap}\mu}{\gamma}$. Dimensionless units will be T for dimensionless time, $H_0=h_0/l_{cap}$, and $\sigma=\varepsilon/l_{cap.}$ This gives the dimensionless equation (9), where a subscript of T or X denotes differentiation with respect to that dimensionless variable.

$$3\sigma_T=3H_0^{\,2}\sigma_X\left(\sigma_X-\sigma_{XXX}\right)+H_0^{\,3}\left(\sigma_{XX}-\sigma_{XXXX}\right) \tag{9}$$

Because σ and its derivatives are much smaller than H_0, this can be linearized as

$$3\sigma_T=H_0^{\,3}\left(\sigma_{XX}-\sigma_{XXXX}\right) \tag{10}$$

The expected solution to this equation is of the form

$$\sigma=Ce^{ikX+sT} \tag{11}$$

Substituting this form into equation (9) gives the characteristic equation for s,

$s=-\frac{H_0^{\ 3}}{3}(k^4+k^2)$. Because s<0, the system is stable; waves decay over time. Because s is a real number, this is not a traveling wave solution; it is film leveling. If dimensional variables are reintroduced, and k is replaced with a dimensional wavenumber, $q=k/l_{cap}$, the solution is the following.

$$\varepsilon = C\exp\left(-\frac{h_0^{\ 3}\gamma q^4}{3\mu}\left(1+\frac{1}{q^2 l_{cap}^{\ \ 2}}\right)t\right)\exp\left(iqx\right)$$

This is the exact same solution that was found earlier, in equation (8). This makes sense because there is no driving force in the system; it is just relaxing from an initial disturbance. If there is a driving force, such as a shear stress on the surface (produced by wind, say) then there should be traveling waves.

Propagation Under Shear Stress

To add a surface shear into the system, one must change the equations used in the lubrication approximation. In deriving equation (4), it was assumed that there was no shear at the free surface. If a shear force, τ, is applied at the free surface, the velocity profile would be

$$u=\frac{1}{\mu}\frac{\partial p}{\partial x}\left(\frac{z^2}{2}-hz\right)+\frac{\tau}{\mu}z \qquad (12)$$

Applying the mass balance (equation (3)) to this new velocity profile gives a new form for the evolution of the film thickness

$$\frac{\partial}{\partial x}\int_0^{h(x)}\left(\frac{1}{\mu}\frac{\partial p}{\partial x}\left(\frac{z^2}{2}-hz\right)+\frac{\tau}{\mu}z\right)dz=-\frac{\partial h}{\partial t}$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}\frac{\partial p}{\partial x}\left(-\frac{h^3}{3}\right)+\frac{\tau}{\mu}\frac{h^2}{2}\right)=-\frac{\partial h}{\partial t}$$

$$\tau h\frac{\partial h}{\partial x}+\frac{h^2}{2}\frac{\partial \tau}{\partial x}-\frac{1}{3}\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)=-\mu\frac{\partial h}{\partial t} \quad (13)$$

If there is wind over the film, there would be a constant shear force. Substituting a constant shear force and pressure as caused by gravitational and capillary effects into equation (13) gives

$$\tau h\frac{\partial h}{\partial x}-\frac{1}{3}\left(3h^2\frac{\partial h}{\partial x}\left(\rho g\frac{\partial h}{\partial x}-\gamma\frac{d^3h}{dx^3}\right)+h^3\left(\rho g\frac{\partial^2 h}{\partial x^2}-\gamma\frac{d^4h}{dx^4}\right)\right)=-\mu\frac{\partial h}{\partial t}$$

Again, using a disturbance $h=h_0+\varepsilon$ and linearizing, the equation reduces to

$$\frac{\tau h_0}{\gamma}\frac{\partial \varepsilon}{\partial x}-\frac{1}{3}h_0^{\ 3}\left(\frac{\rho g}{\gamma}\frac{\partial^2\varepsilon}{\partial x^2}-\frac{d^4\varepsilon}{dx^4}\right)=-\frac{\mu}{\gamma}\frac{\partial \varepsilon}{\partial t}$$

Substituting the expected solution, $\varepsilon=Ce^{ikx+st}$, gives the characteristic equation

$$\mu s=-ik\tau h_0-\frac{h_0^{\ 3}}{3}\left(\gamma k^4+\rho g k^2\right)$$

Because s is complex, this wave travels forward and decays in time. The imaginary part of the exponent in the solution is $k\left(x-\frac{\tau h_0}{\mu}t\right)$, so the wave is moving forward with a speed $V=\frac{\tau h_0}{\mu}$. It is interesting to note that the wave speed is independent of the wave number k, so all waves travel at the same speed. The growth rate of the wave is the real part of s, $G=-\frac{h_0^{\ 3}\gamma}{3\mu}\left(k^4+\frac{k^2}{l_{cap}^{\ 2}}\right)$. Because G is negative, this wave is stable. However, if the disturbance were large enough, there would be growth effects due to wind interacting with a sloped surface. To solve this problem, the small-slope assumption would have to be abandoned, and the horizontal force due to wind would be altered.

Acoustically Driven Wave

If there were an acoustic wave passing over the film, there would be a varying shear force above the film due to the motion of the air. The velocity of the air is proportional to the shear stress on the surface of the film, so the air velocity must be determined. The pressure in a sound wave is $p=p_0+A\sin(\omega t-Kx)$. The velocity of the air is in phase with the pressure, $v=B\sin(\omega t-Kx)$. The values of A and B depend on factors such as the density and compressibility of the air, and the amplitude of the wave. So, the pressure in the fluid is now described as $p - p_{atm} = \rho g h - \gamma \frac{d^2 h}{dx^2} + A\sin(\omega t - Kx) + f(z)$.

Substituting this pressure and a shear force proportional to air velocity into equation (13) gives

$$B\sin(\omega t - Kx) h \frac{\partial h}{\partial x} - \frac{h^2}{2} BK\cos(\omega t - Kx) - \frac{1}{3}\frac{\partial}{\partial x}\left(h^3 \left(\rho g \frac{\partial h}{\partial x} - \gamma \frac{d^3 h}{dx^3} - KA\cos(\omega t - Kx) \right)\right) = -\mu \frac{\partial h}{\partial t}$$

Assuming a small perturbation, $h=h_0+\varepsilon$, and linearizing, the equation becomes

$$B\sin(\omega t - Kx) h_0 \frac{\partial \varepsilon}{\partial x} - \frac{h_0^{\,2}}{2} BK\cos(\omega t - Kx) - \frac{h_0^{\,3}}{3}\left(\rho g \frac{\partial^2 \varepsilon}{\partial x^2} - \gamma \frac{d^4 \varepsilon}{dx^4} - K^2 A\sin(\omega t - Kx) \right) = -\mu \frac{\partial \varepsilon}{\partial t}$$

Rearranging, this equation becomes the driven equation below.

$$B\sin(\omega t - Kx) h_0 \frac{\partial \varepsilon}{\partial x} - \frac{h_0^{\,3}}{3}\left(\rho g \frac{\partial^2 \varepsilon}{\partial x^2} - \gamma \frac{d^4 \varepsilon}{dx^4} \right) + \mu \frac{\partial \varepsilon}{\partial t} = \frac{h_0^{\,2}}{2} BK\cos(\omega t - Kx) - \frac{h_0^{\,3}}{3} K^2 A\sin(\omega t - Kx)$$

If B were negligible, the film would only be affected by the pressure from the sound wave, not the shear from the sound wave. Ignoring terms that are multiplied by B, the equation becomes

$$\frac{h_0^{\,3}}{3}\left(\rho g\frac{\partial^2\varepsilon}{\partial x^2}-\gamma\frac{d^4\varepsilon}{dx^4}\right)-\mu\frac{\partial\varepsilon}{\partial t}=\frac{h_0^{\,3}}{3}K^2 A\sin(\omega t-Kx)$$

The homogeneous solution is of the form $\varepsilon\text{=}Ce^{ikx+st}$ with $-\frac{h_0^{\,3}}{3\mu}\left(\rho g k^2+\gamma k^4\right)=s$ as with the relaxation of a disturbance (equation 8). The particular solution should be of the form Fsin(ωt-Kx)+Gcos(ωt-Kx). Substituting this form in the equation gives a particular solution of

$$\varepsilon_p=A_p\left(3\mu\omega\cos(\omega t-Kx)-h_0^{\,3}K^2\left(\rho g+\gamma\right)\sin(\omega t-Kx)\right)$$
$$A_p=\frac{Ah_0^{\,3}K^2}{K^4h_0^{\,6}(\rho g+\gamma)^2+9\omega^2\mu^2}$$

This is the phase shifted driving pressure. The low point in the film occurs behind the maximum pressure in the sound wave. Adding the homogeneous and particular solutions, the full solution of a thin film under the influence of horizontal acoustic sound waves is

$$\varepsilon\text{=}e^{-\frac{h_0^{\,3}}{3\mu}\left(\rho g k^2+\gamma k^4\right)t}(C_1\cos kx+C_2\sin kx)+A_p\left(3\mu\omega\cos(\omega t-Kx)-h_0^{\,3}K^2\left(\rho g+\gamma\right)\sin(\omega t-Kx)\right)$$

This describes constant surface waves of the same speed and direction (but but shifted backwards) as the sound waves, added to waves caused by other perturbations which relax exponentially in time. This solution is not valid for high frequency sound waves. For a high frequency wave, a fluid element would have to change direction in a short amount of time for the surface waves to have the same frequency as the sound waves.

Changing directions this quickly would likely introduce inertial effects, which were assumed to be negligible in this analysis.

Edge Driven Waves

Another way to drive a wave besides a shear on the surface is by oscillating the film height at a boundary. The steady state solution for a wave that is formed by oscillations at x=0 of $h=h_0+a\sin\omega t$ should be a forward propagating wave that decays in space, $h = h_0 + ae^{-\beta x}\sin(\omega t - kx)$. Dividing lengths by l_{cap} and times by $\frac{l_{cap}\mu}{\gamma}$ gives the dimensionless equation $H = H_0 + Ae^{-BX}\sin(WT - KX)$, where capital letters represent dimensionless properties. H is the dimensionless film thickness, B is the dimensionless spacial decay rate, A is the dimensionless amplitude, W is the dimensionless angular frequency, and K is the dimensionless wave number. Substituting this expression for $\sigma=H-H_0$ into equation (10) gives the following equations

$$2BK(1+2K^2) - 4B^3K - 3\frac{W}{H_0^{\ 3}} = 0$$
$$B^4 + K^4 + K^2 - B^2(1+6K^2) = 0$$

This system is not analytically solvable, but it can be solved graphically. Figure 3 shows curves for the two equations in the range where K and B are both positive. The U-shaped curves represent solutions to the first equation at W/H_0^3=0.01, 0.05, 0.1, 1, and 3, in that order when looking at the curves from left to right. The curves that begin at the origin and at B=1 represent the two positive solutions of the second equation. By trying a wide range of values for W/H_0^3, one can see that each value only corresponds to one set of (K,B); the curve starting at B=1 never intersects any other curves.

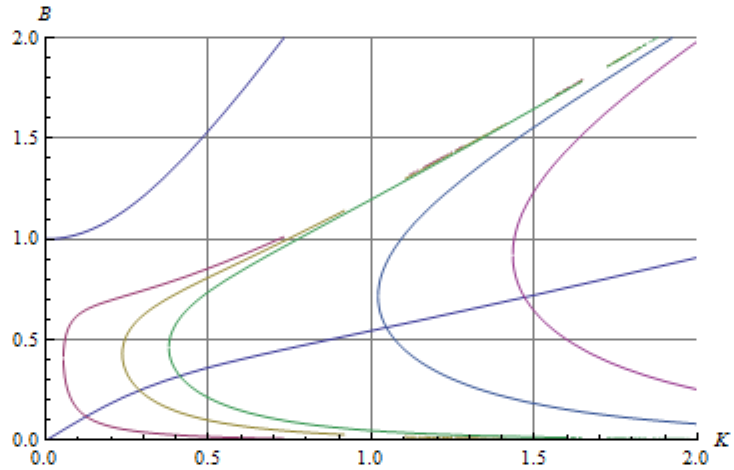


Figure 3: Graphical solutions for parameters in propagating wave

As W/H_0^3 increases, K and B both increase. The phase velocity also depends on W/H_0^3. The phase velocity of a wave is ω/k, or $\frac{W\gamma}{K\mu}$. If this ratio was a constant, then the wave velocity would be a constant for all driving frequencies. However, this is not the case. The values of W/H_0^3 in Figure 4 are 0.05, 0.1, 0.2, and so on, each value twice the previous one. While this is happening, K is not doubling. At first, K increased by a factor of 1.4, but this factor steadily decreased. So W/K is not a constant; W/K increases as W and K increase. This means that if the boundary condition at x=0 were not a pure sinusoid, but was a superposition of different frequencies, the higher frequency waves would move faster than the lower frequency waves, causing the shape of the waveform to change. The higher frequency waves would also decay over a smaller distance because B is larger, so the general waveform would be composed of fewer high frequency components far from the driving source.

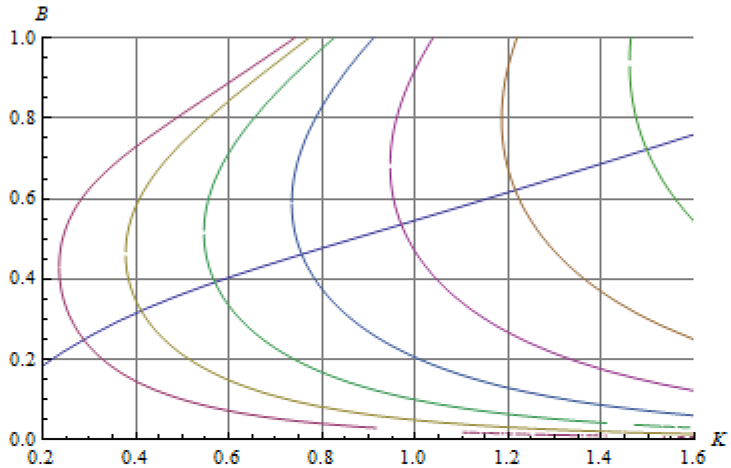
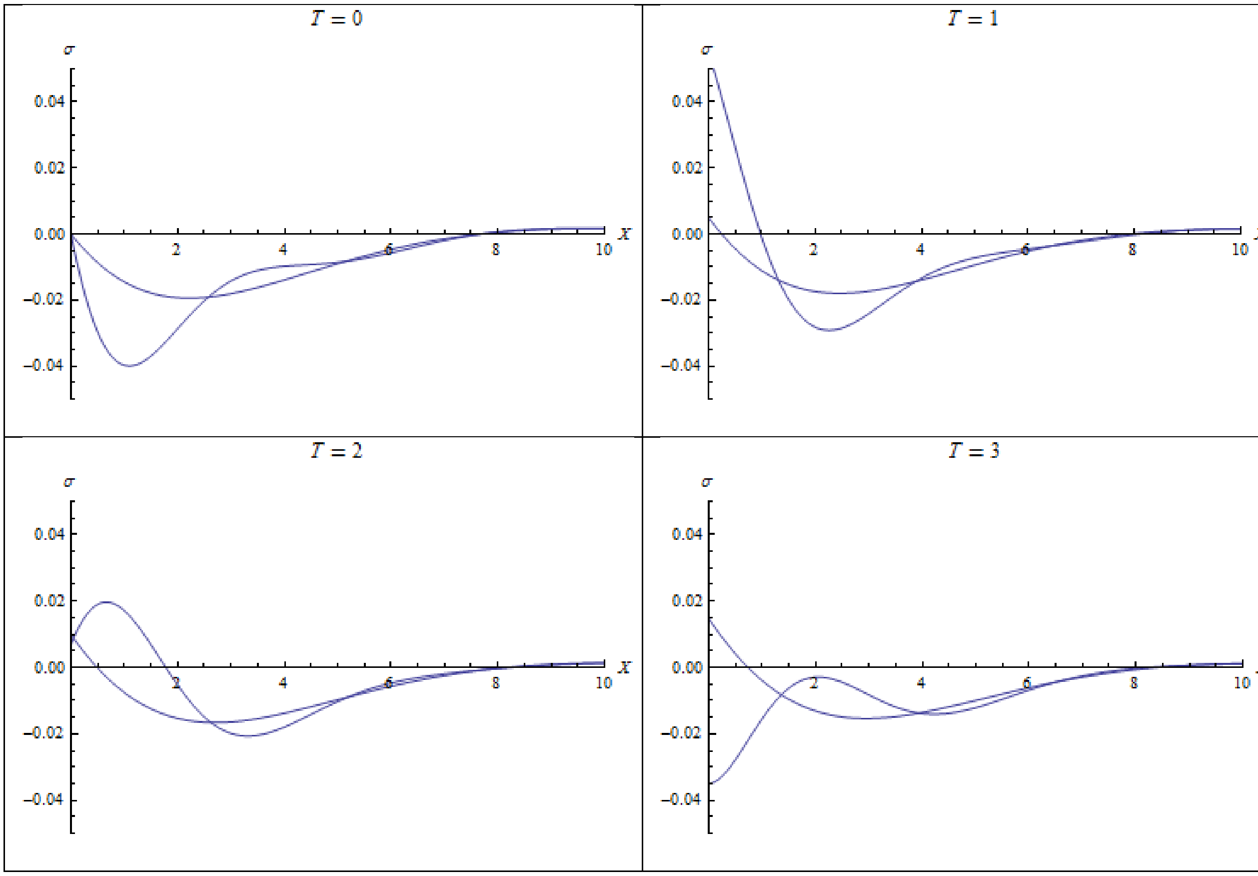


Figure 4: Graphical solutions for parameters in propagating wave, doubling W/H_0^3

Below are plots of the behavior at different times when the boundary condition is $H(X=0)=1+0.05(\sin(0.1T)+\sin(1.6T))$. Using the graphing method, K and B can be found for each value of W, so the solution is

$$\sigma = 0.05(e^{-0.32X}\sin(0.1T-0.41X)+e^{-0.62X}\sin(1.6T-1.22X)).$$

This solution is plotted in Figures 5-8 at different times, along with the solution if there was only a wave for W=0.1. One can see that the high frequency component moves faster than the low frequency component as the trough of the composite wave passes the trough of the low frequency wave (T=1). Also, because the high frequency wave decays in a shorter distance, the total shape is about the same as the shape if there were only a low frequency wave once X>6 .



Figures 5-8: Shape of the surface at various times, responding to two different boundaries

It should be noted that the equations predict waves travelling in both directions, both growing and decaying in space. Figure 9 shows solutions of K and B for different values of W/H_0^3 with every combination of signs for K and B.

When K and B have opposite signs, the wave is growing in the direction of propagation. This unstable solution, although mathematically possible, breaks the assumptions used in finding it, so it cannot exist. In solving for these solutions, it was assumed that the film is thin and the disturbance and its derivatives are much smaller than the average film height. If the disturbance were to grow exponentially, these assumptions would no longer be valid, so the system would not be described in the same way. Also, the lubrication approximation neglects inertial terms. If the wave were to grow very large, fluid would have to move quickly. Inertial effects would resist this acceleration of the fluid, preventing the growth of waves. Lastly, if waves grew large enough that the amplitude was comparable to the average thickness, holes would form in the film, and the behavior would change greatly.

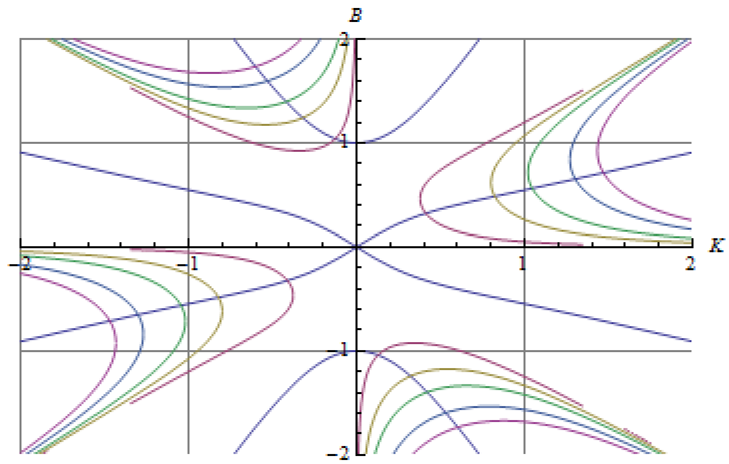


Figure 9: Graphical solutions for parameters of propagating waves, all quadrants.

When K and B are both negative, the wave is travelling in the negative X direction and decaying in that direction. By combining waves moving in both directions, a reflecting wave can be described. At X=0, the boundary is again $\sigma=Ae^{-BX}\sin(WT-KX)$. Where the wave reflects at X=L, the reflected wave, σ_r, must be equal to the incident wave, so $\sigma_r(X=L)=Ae^{-BL}\sin(WT-KL)$. With this boundary condition, the reflected wave must be $\sigma_r=Ae^{-B(2L-X)}\sin(WT-K(2L-X))$. Combining σ_r with σ gives the equation of a reflecting wave. When plotted, one can barely tell the difference between σ and $\sigma+\sigma_r$. This is because the wave decays so much by the time it reaches the wall, there is not much to reflect. In other words, e^{-BL} is small. Because the reflected wave has such a low amplitude, a standing wave cannot be created. There is not enough interference from the reflected wave to create nodes and antinodes.

Conclusions and Future Work

The behavior of thin films can be described by the lubrication approximation. This approximation was used to find the behavior of small disturbances as they evolve in space and time. Throughout the derivations, many assumptions were made. To achieve more accurate descriptions of behavior, the nonlinear differential equation describing waves in the lubrication approximation (equation 9) should be solved. In order to extend the results to faster waves and deeper liquid, inertial terms should be included. By doing this, one could check that the behavior approaches the well known behavior of waves in deeper liquids.

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