PHYS 551 Homework 4 Solutions

Problem 2

$$\begin{aligned}
|\psi\rangle &= \cos\theta\,|J,J\rangle + \sin\theta\,|J,-J\rangle \\
\langle J_z\rangle &= \langle\psi|\,J_z\,|\psi\rangle = (\cos\theta\,\langle J,J| + \sin\theta\,\langle J,-J|)(J\cos\theta\,|J,J\rangle - J\sin\theta\,|J,-J\rangle) = J\cos(2\theta) \\
\Delta J_x &= \sqrt{\langle J_x^2\rangle - \langle J_x\rangle^2} \\
\langle J_x\rangle &= \frac{1}{2}\langle\psi|\,J_+ + J_-\,|\psi\rangle = \sqrt{\frac{J}{2}}(\cos\theta\,\langle J,J| + \sin\theta\,\langle J,-J|)(\cos\theta\,|J,J-1\rangle + \sin\theta\,|J,1-J\rangle) \\
&= \begin{cases} 0 & J > 1/2 \\ \frac{\sin 2\theta}{2} & J = 1/2 \end{cases} \\
\left\langle J_x^2\right\rangle &= \frac{1}{4}\langle\psi|\,(J_+ + J_-)^2\,|\psi\rangle = \frac{J}{2}(\cos\theta\,\langle J,J| + \sin\theta\,\langle J,-J|)\times \\
&\quad [\cos\theta\,|J,J\rangle + \sin\theta\,|J,-J\rangle + \sqrt{\frac{2J-1}{J}}(\cos\theta\,|J,J-2\rangle + \sin\theta\,|J,2-J\rangle)] \\
\left\langle J_x^2\right\rangle &= \begin{cases} \frac{J}{2} & J > 1 \\ \frac{1}{2}(1+\sin 2\theta) & J = 1 \\ \frac{1}{4} & J = 1/2 \end{cases} \\
\Delta J_x &= \begin{cases} \sqrt{\frac{J}{2}} & J > 1 \\ \frac{1}{\sqrt{2}}(1+\sin 2\theta)^{1/2} & J = 1 \\ \frac{\cos 2\theta}{2} & J = 1/2 \end{cases}
\end{aligned}$$

$$\begin{aligned}
\langle J_y\rangle &= \frac{1}{2i}\langle\psi|\,J_+ - J_-\,|\psi\rangle = i\sqrt{\frac{J}{2}}(\cos\theta\,\langle J,J| + \sin\theta\,\langle J,-J|)(\cos\theta\,|J,J-1\rangle - \sin\theta\,|J,1-J\rangle) = 0 \\
\left\langle J_y^2\right\rangle &= -\frac{1}{4}\langle\psi|\,(J_+ - J_-)^2\,|\psi\rangle = \frac{J}{2}(\cos\theta\,\langle J,J| + \sin\theta\,\langle J,-J|)\times \\
&\quad [\cos\theta\,|J,J\rangle + \sin\theta\,|J,-J\rangle - \sqrt{\frac{2J-1}{J}}(\cos\theta\,|J,J-2\rangle + \sin\theta\,|J,2-J\rangle)] \\
\left\langle J_y^2\right\rangle &= \begin{cases} \frac{J}{2} & J > 1 \\ \frac{1}{2}(1-\sin 2\theta) & J = 1 \\ \frac{1}{4} & J = 1/2 \end{cases} \\
\Delta J_y &= \begin{cases} \sqrt{\frac{J}{2}} & J > 1 \\ \frac{1}{\sqrt{2}}(1-\sin 2\theta)^{1/2} & J = 1 \\ \frac{1}{2} & J = 1/2 \end{cases}
\end{aligned}$$

Now consider separately:

For $J = 1/2 : \Delta J_x = \cos(2\theta)/2 < \sqrt{|\langle J_z\rangle|/2} = \cos^{1/2}(2\theta)/2$ for all $\theta \neq n\pi/2$, and $\Delta J_x \Delta J_y = \cos(2\theta)/4 = \langle J_z\rangle/2$

For $J = 1 : \Delta J_y = \frac{1}{\sqrt{2}}(1-\sin 2\theta)^{1/2} < \sqrt{|\langle J_z\rangle|/2} = \frac{1}{\sqrt{2}}\cos^{1/2}(2\theta)$ for $0 < \theta < \pi/2$

$\Delta J_x = \frac{1}{\sqrt{2}}(1 + \sin 2\theta)^{1/2} < \sqrt{|\langle J_z \rangle|/2} = \frac{1}{\sqrt{2}} \cos^{1/2}(2\theta)$ for $\pi/2 < \theta < \pi$

$\Delta J_x \Delta J_y = \frac{1}{2}\sqrt{1 - \sin^2 1} = \frac{1}{2} \cos 2\theta = \langle J_z \rangle /2$

For $J > 1 : \Delta J_x = \Delta J_y = \sqrt{J/2} \geq \sqrt{(J/2) \cos 2\theta}; \Delta J_x \Delta J_y = J/2 \geq (J/2) \cos 2\theta$

Another way to measure squeezing is to define the uncertainty in the measurement of the precession angle $\Delta\alpha_x = \Delta J_x/|\langle J_z \rangle|$. For $J = 1/2$ we get $\Delta\alpha_x = 1$, no improvement in the precision of angle measurement can be achieved by changing θ. For $J = 1$, $\Delta\alpha_y = \frac{1}{\sqrt{2}}(1 - \sin 2\theta)^{1/2}/\cos 2\theta$, which reaches minimum $\Delta\alpha_y = 1/2$ for $\theta = \pi/4$. Hence if the $J = 1$ system is made from two $S = 1/2$ particles, the uncertainty in the angle measurement is improved by a factor of 2, not $\sqrt{2}$, indicating "Heisenberg" uncertainty. For higher spin systems this simple form does not work, in fact there is no general explicit form of a wavefunction that achieves Heisenberg uncertainty, but it can be constructed for specific J.