Appendix A: Proofs

Proof of Lemma 1. Consider two experts A and B with net worth's n_t^A and n_t^B, respectively. Denote by u_t^A and u_t^B the maximal expected utilities that these experts can get in equilibrium from time t onwards. We need to show that $u_t^A/n_t^A = u_t^B/n_t^B$. Suppose not, e.g. $u_t^A/n_t^A > u_t^B/n_t^B$. Denote by $k_s^A, dc_s^A, \varphi_s^A; s \geq t$ the optimal dynamic strategy of expert A, which attains utility u_t^A, i.e.

$$u_t^A = E_t\left[\int_t^\infty e^{-\rho(s-t)} dc_{t+s}^A\right].$$

Because the strategy is feasible, the process

$$dn_s^A = rn_s^A ds + (k_s^A q_s)\left[\left(\frac{a-\iota(q_s)}{q_s} + g(q_s) + \mu_s^q + \sigma\sigma_s^q - r\right) ds + \varphi_s^A(\sigma+\sigma_s^q)dZ_s\right] - dc_s^A.$$

stays nonnegative. Let $\varsigma = n_t^B/n_t^A$, and consider the strategy $\varsigma k_s^A, \varsigma dc_s^A, \varphi_s^A; s \geq t$ of expert B. This strategy is also feasible, because it leads to a non-negative wealth process $n_t^B = \varsigma n_t^A$, and it delivers the expected utility of ςu_t^A to expert B. Thus, $u_t^B \geq \varsigma u_t^A$, leading to a contradiction.

Therefore, for all experts their expected utility under the optimal trading strategy is proportional to wealth. It follows that $\theta_t = u_t^A/n_t^A = u_t^B/n_t^B$. ■

Proof of Lemma 2. First, assume that the process $\theta_t, t \geq 0$ represents marginal value of experts' net worth. Let us show that then θ_t must satisfy the Bellman equation (5), which characterizes the experts' optimal strategies, and the transversality condition. Let $\{k_t \geq 0, dc_t \geq 0, \varphi_t \geq \tilde{\varphi}\}$ be an arbitrary admissible strategy (i.e. does not violate the solvency constraint). We argue that the process

$$\Theta_t = \int_0^t e^{-\rho s} dc_s + e^{-\rho t}\theta_t n_t$$

is always a supermartingale; and it is a martingale if the strategy $\{k_t, dc_t, \varphi_t\}$ is optimal. Note that the maximal payoff that an expert can obtain at time t is

$$\theta_t n_t \geq E_t\left[\int_t^{t+s} e^{-\rho(s'-t)}\, dc_{s'} + e^{-\rho s}\theta_{t+s}n_{t+s}\right],$$

where equality is attained if the agent follows an optimal strategy from time t to $t+s$, since $\theta_{t+s}n_{t+s}$ is the maximal payoff that the agent can attain from time $t+s$ onwards. Therefore,

$$\Theta_t = \int_0^t e^{-\rho s'} dc_{s'} + e^{-\rho t}\theta_t n_t \geq E_t\left[\int_0^{t+s} e^{-\rho s'} dc_{s'} + e^{-\rho s} n_{t+s}\theta_{t+s}\right] = E_t[\Theta_{t+s}]$$

with equality if the agent follows the optimal strategy.

Differentiating Θ_t with respect to t using Ito's lemma, we find

$$d\Theta_t = e^{-\rho t}(dc_t - \rho\theta_t n_t\, dt + d(\theta_t n_t))$$

For the optimal strategy we have $E[dc_t - \rho\theta_t n_t\, dt + d(\theta_t n_t)] = 0$ (since Θ_t is a martingale), and for any arbitrary strategy we have $E[dc_t - \rho\theta_t n_t\, dt + d(\theta_t n_t)] \leq 0$ (since $\Theta_t$is a supermartingale). Therefore, the optimal strategy of any expert is characterized by the Bellman equation (5). To verify that the transversality condition holds under an optimal strategy k_t, c_t, φ_t, note that (a) the expected payoff of an expert with net worth n_t is given by

$$\theta_0 n_0 = E\left[\int_0^\infty e^{-\rho s} dc_s\right] = \lim_{t\to\infty} E\left[\int_0^t e^{-\rho s} dc_s\right],$$

where expectation value and limit can be interchanged by the Monotone Convergence Theorem (because $dc_s \geq 0$), and (b) for all t,

$$\theta_0 n_0 = E\left[\int_0^t e^{-\rho s} dc_s + e^{-\rho t}\theta_t n_t\right].$$

Taking $t \to \infty$ in the latter formula, and combining with the former, we get the transversality condition.

Conversely, let us show that if a process θ_t satisfies the Bellman equation and the transversality condition holds, then θ_t represents the experts' marginal value of net worth and characterizes their optimal strategies. Note that, as we just demonstrated, equation (5) implies that the process Θ_t is always a supermartingale, and a martingale for any strategy $\{k_t, dc_t, \varphi_t\}$ that attains the maximum in equation (5). Thus, any expert who follows such a strategy attains the payoff of

$$E\left[\int_0^\infty e^{-\rho s}\, dc_s\right] = \lim_{t\to\infty} E\left[\int_0^t e^{-\rho s}\, dc_s\right] = \lim_{t\to\infty}(\theta_0 n_0 - E[e^{-\rho t}\theta_t n_t]) = \theta_0 n_0$$

where the last equality follows from the transversality condition.

Any alternative strategy achieves utility

$$\lim_{t\to\infty} E\left[\int_0^t e^{-\rho s} dc_s\right] \leq \lim_{t\to\infty}(\theta_0 n_0 - E[e^{-\rho t}\theta_t n_t]) \leq \theta_0 n_0$$

where the last inequality holds because $\theta_t n_t \geq 0$. We conclude that $\theta_0 n_0$ is the maximal utility that any expert with net worth n_0 can attain, and that the optimal strategy must solve the maximization problem in the Bellman equation. ■

Proof of Lemma 3. Aggregating over all experts, the law of motion of N_t is

$$dN_t = rN_t\, dt + \psi_t(K_t q_t)[(E_t[r_t^k] - r)\, dt + \varphi_t(\sigma + \sigma_t^q)\, dZ_t] - dC_t,$$

where C_t is are aggregate payouts, and the law of motion of K_t is

$$dK_t = (\psi_t g(q_t) - (1-\psi_t)\underline{\delta})K_t\, dt + \sigma K_t\, dZ_t$$

$$d(1/K_t) = -(\psi_t g(q_t) - (1-\psi_t)\underline{\delta})(1/K_t)\, dt + \sigma^2(1/K_t)\, dt - \sigma(1/K_t)\, dZ_t.$$

Combining the two equations, and using Ito's lemma, we get

$$\begin{aligned} d\eta_t \;\; &= \;\; \left(r - \psi_t g(q_t) + (1-\psi_t)\underline{\delta} + \sigma^2\right)\eta_t\, dt + \psi_t q_t(E_t[r_t^k] - r)\, dt - \psi_t\varphi_t q_t\sigma(\sigma+\sigma_t^q)\, dt \\ &\quad + (\psi_t q_t\varphi_t(\sigma+\sigma_t^q) - \sigma\eta_t)\, dZ_t - d\zeta_t \end{aligned}$$

Substituting $\sigma_t^\eta = \psi_t\varphi_t q_t(\sigma+\sigma_t^q)/\eta_t - \sigma$ into the expression for the drift of η_t, we get (6). Furthermore, if $\sigma_t^q \geq 0$, $\sigma_t^\theta \leq 0$ and $\psi_t > 0$, then Proposition 1 implies that $\varphi_t = \tilde{\varphi}$, $E[r_t^k] - r = -\tilde{\varphi}\sigma_t^\theta(\sigma+\sigma_t^q)$, and so

$$\mu_t^\eta = r - \psi_t g(q_t) + (1-\psi_t)\underline{\delta} - \sigma_t^\eta(\sigma+\sigma_t^\theta) - \sigma\sigma_t^\theta$$

■

Proof of Proposition 6. First, it is suboptimal to employ monitoring the event that the value of the firm's assets falls by less than $n_{t-}/\tilde{\varphi}$ due to a jump. It is possible to guarantee that jumps of size $n_{t-}/\tilde{\varphi}$ or less are never caused by benefits extraction by subtracting value $\tilde{\varphi}k_t|dJ_t^i|$ from the expert's inside equity stake in the event that such a jump occurs. Such an incentive mechanism is costless, since the expert is risk-neutral with respect to jump risks as they are uncorrelated with the experts' marginal value of net worth θ_t. At the same time, monitoring carries the deadweight loss of a verification cost. Also, monitoring is not an effective way to prevent continuous diversion of private benefits, because outside investors have to pay a positive cost of monitoring in response to a possible infinitesimal deviation (recall that we disallow randomized monitoring).

Second, monitoring has to be employed in the event that the value of the firm's assets falls by more than $n_{t-}/\tilde{\varphi}$, because it is the only way to prevent benefit extraction in such large quantities (other than simply keeping the value of the assets below $n_{t-}/\tilde{\varphi}$). Without loss of generality, we can consider contracts that leave the expert with zero net worth if he is caught diverting such large amounts for private benefit. In the event that a loss of size more than $n_{t-}/\tilde{\varphi}$ is verified to have occurred without benefit extraction, recovered capital can be split arbitrarily between the expert and outside investors in an optimal contract. Because the expert is risk-neutral with respect to idiosyncratic risks uncorrelated with aggregate shocks, without loss of generality we can assume that all recovered capital goes to outside debt holders.[24]

In this case, to compensate outside investors for monitoring costs and for the expected value lost in possible default (i.e. event when costly state verification is triggered), expert's net worth has to evolve according to

$$\begin{aligned} dn_t &= rn_t\,dt + (k_t q_t)\Bigg[\left(\frac{a-\iota(q_t)}{q_t} + g(q_t) + \mu_t^q + \sigma\sigma_t^q - r - L(\vartheta_t) - C(\vartheta_t)\right)dt \\ &\quad + dJ_t^i + \tilde{\varphi}(\sigma + \sigma_t^q)\,dZ_t\Bigg] - dc_t, \end{aligned}$$

where $\vartheta_t = 1 - n_t/(\tilde{\varphi}q_t k_t)$ is the expert's debt to total asset ratio (leverage). We set $\varphi_t = \tilde{\varphi}$ to minimize the expert's exposure to aggregate risk, because we assumed that $\sigma_t^q \geq 0$ and $\sigma_t^\theta \leq 0$. ■

Appendix B: Contracting on k_t

Appendix B analyzes the case in which contracting directly on k_t is possible instead of $k_t q_t$. An expert manages capital that follows

$$dk_t = (\Phi(\iota_t) - \delta - b_t)\,k_t\,dt + \sigma k_t\,dZ_t.$$

where b_t is the rate of private benefit extraction, and produces output $(a - \iota_t)dt$. Furthermore, suppose that the expert can get the marginal benefit of $\tilde{\varphi} \leq 1$ units of capital per unit diverted. Denote by q_t the market price of capital, by θ_t, the value of expert funds per dollar, by $\iota(q_t)$ the optimal level of investment and by $g(q_t) = \Phi\left(\iota(q_t)\right) - \delta$ the implied growth rate. What is the optimal contract, if k_t rather than $k_t q_t$ is used as the measure of performance? In this section we follow the literature on dynamic contracting to derive the implications of contracting directly on k_t, e.g. see DeMarzo and Sannikov (2006).

[24] There are other optimal contracts, for example the expert could be fully insured against drops in asset value that are verified to involve no benefit extraction. Of course, the expert would have to pay a 'premium' for such insurance in the event that there were no jump losses.

Consider contracts based on the agent's net worth as a state variable. The "official" net worth follows

$$dn_t = rn_t\ dt + \beta_t\left(dk_t - g(q_t)k_t\ dt\right) - \sigma_t^\theta \beta_t \sigma k_t\ dt, \tag{8}$$

and the agent also gets funds at rate $\tilde{\varphi} b_t q_t$ if he extracts benefits $b_t \geq 0$. The incentive constraint is

$$\beta_t \geq \tilde{\varphi} q_t$$

since the expert gets $\tilde{\varphi} q_t$ units of net worth (that can be used elsewhere to gain the utility of $\tilde{\varphi} q_t \theta_t$) for one unit of capital diverted. Note that the stochastic as well as the deterministic portion of the law of motion of n_t depends directly on k_t, so households need to observe k_t directly in order to write a contract that rewards the expert according to equation (8).

Note that $e^{-\rho t}\theta_t n_t$ is a martingale when the expert refrains from extracting benefits and does not consume. We have

$$\begin{aligned}
d(\theta_t n_t) &= \theta_t(rn_t\ dt + \beta_t \sigma k_t\ dZ_t - \sigma_t^\theta \beta_t \sigma k_t\ dt) + (\mu_t^\theta\ dt + \sigma_t^\theta\ dZ_t)\theta_t n_t + \sigma_t^\theta \theta_t \beta_t \sigma k_t\ dt \\
&= \theta_t(rn_t\ dt + \beta_t \sigma k_t\ dZ_t) + \left((\rho - r)\ dt + \sigma_t^\theta\ dZ_t\right)\theta_t n_t \\
&= \rho(\theta_t n_t)\ dt + \text{volatility term},
\end{aligned}$$

where we use as in Section 3 the property that $\mu_t^\theta = (\rho - r)$.

Next, we study the price of capital, q_t. We derive a pricing equation for capital by setting the expected return that households earn from investing in capital to r.

If contracting is based on k_t only, then households hire experts to manage their capital, but households themselves take on the price risk. The market price of capital still depends on the experts' risk-taking capacity. The return that households earn on their capital holdings k_t is given by

$$\begin{aligned}
(k_t q_t)\, \underline{r}_t^k &= (a - \iota(q_t))\, k_t\ dt + d\,(q_t k_t) - \beta_t k_t \sigma\ dZ_t + \beta_t \sigma_t^\theta \sigma k_t\ dt \\
&= (a - \iota\,(q_t))\, k_t\ dt + (q_t k_t)\left[(\mu_t^q + g\,(q_t) + \sigma\sigma_t^q)\ dt + (\sigma + \sigma_t^q)\ dZ_t\right] \\
&\quad - \beta_t k_t \sigma\ dZ_t + \beta_t \sigma_t^\theta \sigma k_t\ dt
\end{aligned}$$

If $\sigma_t^\theta < 0$, then households optimally set $\beta_t = \tilde{\varphi} q_t$ to minimize the costs of compensating experts for risk. In expectation $\underline{r}_t^k$ should equal r, so we need

$$\frac{a - \iota\,(q_t)}{q_t} + \mu_t^q + g(q_t) + \sigma\sigma_t^q - r + \tilde{\varphi}\sigma_t^\theta \sigma = 0.$$

This equation is different from the pricing equation (EK) because the risk premium is based only on exogenous risk (for which households must compensate the experts that manage their capital).

Also, the law of motion of η_t will be different. Combining the law of motion of n_t and the condition that the households must get an expected return of r, we get the equation

$$dn_t = rn_t\ dt + (k_t q_t)\left[\left(\frac{a - \iota(q_t)}{q_t} + \mu_t^q + g(q_t) + \sigma\sigma_t^q - r\right) dt + \tilde{\varphi}\sigma\ dZ_t\right] - dc_t,$$

which does not have the endogenous risk term. As a result,

$$dN_t = rN_t\ dt + \psi_t\,(K_t q_t)\left[\left(\frac{a - \iota(q_t)}{q_t} + \mu_t^q + g(q_t) + \sigma\sigma_t^q - r\right) dt + \tilde{\varphi}\sigma\ dZ_t\right] - dC_t.$$

Because $dK_t/K_t = (\psi_t g(q_t) - (1-\psi_t)\underline{\delta})\ dt + \sigma\ dZ_t$, and so

$$d(1/K_t)/(1/K_t) = -(\psi_t g(q_t) - (1-\psi_t)\underline{\delta})\ dt + \sigma^2\ dt - \sigma\ dZ_t,$$

we get

$$\begin{aligned} d\eta_t &= \left(r - \psi_t g(q_t) + (1-\psi_t)\underline{\delta} + \sigma^2\right)\eta_t\, dt \\ &\quad + \psi_t q_t \left(\frac{a-\iota(q_t)}{q_t} + \mu_t^q + g(q_t) + \sigma\sigma_t^q - r\right)\ dt \\ &\quad - \psi_t\tilde{\varphi} q_t \sigma^2\ dt + (\psi_t\tilde{\varphi} q_t - \eta_t)\,\sigma\, dZ_t - d\zeta_t. \end{aligned}$$

The volatilities of η_t and q_t are found to be

$$\sigma_t^\eta = \left(\frac{\psi_t\tilde{\varphi} q_t}{\eta_t} - 1\right)\sigma \quad \text{and} \quad \sigma_t^q = \frac{q'(\eta_t)}{q_t}\sigma_t^\eta \eta_t,$$

so there is still amplification through leverage, but no more feedback effect through prices.

Appendix C. Stationary Distribution

Suppose that X_t is a stochastic process that evolve on the state space $[x_L, x_R]$ according to the equation

$$dX_t = \mu^x(X_t)\, dt + \sigma^x(X_t)\, dZ_t \tag{9}$$

If at time $t=0$, X_t is distributed according to the density $d(x,0)$, then the density of X_t at all future dates $t \geq 0$ is described by the forward Kolmogorov equations:

$$\frac{\partial}{\partial t} d(x,t) = -\frac{\partial}{\partial x}\left(\mu^x(x)\, d(x,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\sigma^x(x)^2\, d(x,t)\right).$$

If one of the endpoints is a reflecting barrier, then the boundary condition at that point is

$$-\mu^x(x) d(x,t) + \frac{1}{2}\frac{\partial}{\partial x}(\sigma^x(x)^2 d(x,t)) = 0.$$

A *stationary density* stays fixed over time under the law of motion of the process, so the left-hand side of the Kolmogorov forward equation is $\frac{\partial d(x,t)}{\partial t} = 0$. If one of the endpoints of the interval $[x_L, x_R]$ is reflecting, then integrating with respect to x and using the boundary condition at the reflecting barrier to pin down the integration constant, we find that the stationary density is characterized by the first-order ordinary differential equation

$$-\mu^x(x) d(x) + \frac{1}{2}\frac{\partial}{\partial x}(\sigma^x(x)^2 d(x)) = 0.$$

To compute the stationary density numerically, it is convenient to work with the function $D(x) = \sigma^x(x)^2 d(x)$, which satisfies the ODE

$$D'(x) = 2\frac{\mu^x(x)}{\sigma^x(x)^2} D(x). \tag{10}$$

Then $d(x)$ can be found from $D(x)$ using $d(x) = \frac{D(x)}{\sigma^x(x)^2}$.

With absorbing boundaries, the process eventually ends up absorbed (and so the stationary distribution is degenerate) unless the law of motion prevents (9) it from hitting the boundary with probability one. A non-degenerate stationary density exists with an absorbing boundary at x_L if the boundary condition $D(x_L) = 0$ can be satisfied together with $D(x_0) > 0$ for $x_0 > x_L$. For this to happen, we need

$$\log D(x) = \log D(x_0) - \int_x^{x_0} \frac{2\mu^x(x')}{\sigma^x(x')^2}\, dx' \to -\infty, \text{ as } x \to x_L$$

i.e $\int_{x_L}^{x_0} \frac{2\mu^x(x)}{\sigma^x(x)^2}\, dx = \infty$. This condition is satisfied whenever the drift that pushes X_t away from the boundary x_L (so we need $\mu^x(x) > 0$) is strong enough working against the volatility that may move X_t towards x_L. For example, if X_t behaves as a geometric Brownian motion near the boundary $x_L = 0$, i.e. $\mu^x(x) = \mu x$ and $\sigma^x(x) = \sigma x$, with $\mu > 0$, then $\int_0^{x_0} \frac{2\mu^x(x)}{\sigma^x(x)^2}\, dx = \int_0^{x_0} \frac{2\mu}{\sigma^2 x}\, dx = \infty$.