Funding Horizons, Interest Rates, and Growth*

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Abstract

How to finance investment when continual effort of key workers is essential for developing and maintaining intangibles? If key workers (engineers) cannot precommit to provide effort in the future, how does this affect the horizons of external funding? We develop a model of funding horizons in which, the further distant into the future, the larger the fraction of revenue attributed to engineers' *cumulative* efforts. Looking ahead from the time of investment, we show that because engineers cannot precommit to work for less than their marginal contribution to future production, more surplus goes to them in the distant future – and concomitantly less goes to external financiers. Hence, ex ante, the external funding capacity of an investing engineer is largely governed by revenues in the near horizon. This limits external finance, leading to possible under-investment. We examine how funding horizons interact with plant dynamics and the aggregate economy. We show that an unanticipated permanent fall in the interest rate in a small open economy can lead to a temporary boom followed by slower growth in the long run.

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1 Introduction

Recently intangible capital appears to have become more important for production and valuation relative to tangible capital.[1] Intangible capital is unique because continual effort of key workers is often essential for developing and maintaining intangibles. Nonetheless the key workers often cannot pecommit to provide effort in the future, because they can work elsewhere. How to finance investment? How does the limited commitment of key workers affect the horizons of external funding? How do funding horizons interact with plant dynamics and the aggregate economy?

To address these questions, we develop a model of funding horizons in which the human capital of an entrepreneur-cum-engineer is essential for the construction and then maintenance of a production facility. To get a flavour of our model, think of an engineer investing in goods and a building in which to construct plant. There is no obstacle for the engineer to raising funds against the plant: it can be freely sold at the time of investment.[2] What cannot be sold is the engineer's expertise, her human capital, which we assume is acquired through learning-by-doing associated with her gross investment. This inalienability constrains the engineer's fund-raising ability, so the scale of her investment is limited by her net worth.[3]

A saver who buys the plant will subsequently need an engineer's expertise to maintain its productivity. Without adequate maintenance, productivity would deteriorate. Generally we adopt a form of 'roundabout' technology, inspired by Böhm-Bawerk (1889): we suppose that tomorrow's plant productivity is a function of both today's productivity and today's engineering input. Hence, although on any given date the plant's current productivity is historically given, its long-run future productivity will mostly depend upon the forthcoming cumulative maintenance effort. We take an extreme example in this Introduction: with engineer's continual maintenance, each unit of plant yields the deterministic flow returns, $y_{t+1},\ y_{t+2},\ y_{t+3,...}$. If the maintenance is missed once, the plant stops yielding returns forever.

[1] See, for example, Brynjolfsson, Rock and Syverson (2021), Corrado, Hulten, and Sichel (2009), Crouzet, Eberly, Eisfeldt and Papanikolaou (2022), Eisfeldt and Papanikolaou (2013), Karabarbounis and Nieman (2019), McGrattan and Prescott (2010), and Peters and Taylor (2017).

[2] By "sold", we mean handing over all the control rights over the plant, all the equity.

[3] Hart and Moore (1994).

Crucially, because her human capital is inalienable, the engineer cannot commit at the time of investment, ex ante, subsequently to supply her maintenance services to the saver. We assume the match between engineers and plant owners can change over time. In the future, the plant owner can hire any engineer to maintain his plant and the engineer can work for any plant owner. In this period, however, there is only one engineer who can maintain a particular plant – the match is one engineer to one plant owner.

In this Introduction, let us consider a date t, when an owner and an engineer bilaterally bargain over the engineer's "wage" w_t. The surplus to the engineer equals w_t because she can work for another plant from the next period if the bargaining breaks down. The surplus for the plant owner is more substantial, because failure to agree will destroy the future of the plant. Let $R > 1$ be the gross interest rate per period, and V_{t+1} be the continuation value of the plant at the beginning of t+1. The surplus to the owner equals the present value of the continuation value minus the present wage to engineer, $\frac{1}{R}V_{t+1} - w_t$. Let θ and $1-\theta$ be the bargaining powers of the owner and the engineer. We consider the wage is determined to maximize the Nash product,

$$\underset{w_t}{Max} \left(\frac{1}{R} V_{t+1} - w_t \right)^{\theta} (w_t)^{1-\theta}.$$

Through this bargaining, the engineer receives a fraction $1-\theta$ of the continuation value of the plant as wage,

$$w_t = (1-\theta) \frac{1}{R} V_{t+1}.$$

The value to the plant owner equals the present profit (output minus wage) plus the present value of continuation value as

$$V_t = y_t - w_t + \frac{1}{R} V_{t+1}.$$

Then substituting the equilibrium wage, the owner's value equals the present output plus a fraction θ of the present value of the continuation value as

$$V_t = y_t + \frac{\theta}{R} V_{t+1}.$$

Iterating this forward, we learn

$$V_t = y_t + \frac{\theta}{R} y_{t+1} + \left(\frac{\theta}{R} \right)^2 y_{t+2} + \left(\frac{\theta}{R} \right)^3 y_{t+3} + \ \dots \ .$$

Because $\theta < 1$, the plant owner derives the value largely from near-future revenue, say in the next several years.

At the time of investment at t, the engineer raises external fund by selling the plant which starts yielding output from date t+1 at price

$$b = \frac{1}{R}V_{t+1} = \frac{1}{R}y_{t+1} + \frac{\theta}{R^2}y_{t+2} + \frac{\theta^2}{R^3}y_{t+3} + \frac{\theta^3}{R^4}y_{t+4} + \ \dots \ .$$

The horizon for the investing engineer to raise the external fund is relatively short, because the saver who buys the plant to provide funds derives the value largely from near-future revenue. It can thus be said that the external finance is largely near-term even when the plant is long-term.

Present wage to the engineer reflects the long-run influence on future production as

$$w_t = \frac{1-\theta}{R}V_{t+1} = \frac{1-\theta}{R}y_{t+1} + \frac{1-\theta}{R^2}\theta y_{t+2} + \frac{1-\theta}{R^3}\theta^2 y_{t+3} + \ \dots \ .$$

The engineer's wage is forward looking because the engineer's maintenance affects the entire future production.

Parenthetically, we note that an often made criticism of a capitalist economy is that the horizons of shareholders and managers are too short-term. This idea finds an echo in our model. Plant owners – who are akin to shareholders and managers – derive value mainly from the plant's near-term revenues, insofar as they are obliged to pay engineers – their key workers – a forward-looking reward to maintain the plant.[4] The empirical corporate finance literature documents that entrepreneurs usually raise external funding against their future cash flows as well as put up their fixed assets as collateral[5]. The funds raised are typically only a few years' worth of revenue.[6]

Here, however, we are not arguing that the plant is not efficiently maintained after investment. What we are arguing is that there can be *under-investment* ex ante because the external funding horizon is short. At the

[4] De la O and Myers (2021) find that expectations of cash flow growth in the near future explain most movements in the S&P 500 price-dividend and price-earnings ratios.

[5] Lian and Ma (2021) examine firm-level data of US non-financial corporations to document that approximately 80% of corporate borrowing is backed by future cash flow and only 20% by collateral assets – though this 4:1 ratio tends to be lower for smaller US firms and lower in other countries like Japan. Drechsel (2020) also documents that much of US corporate borrowing is constrained by firms' current earnings.

[6] Lian and Ma (2021) further examine debt covenants to find that, at the 25 percentile, the cash-flow-based debt is restricted by 3 years' worth of EBITDA (earnings before interest, taxes, depreciation and amortization) and, at the 75 percentile, by 4.5 years' worth.

time of investment, the engineer uses her net worth (after consumption) and the external fund to finance the total investment cost: the unit investment cost times the investment scale. Thus aggregate investment scale is limited by

$$\text{investment scale } = \frac{\text{aggregate net worth of engineers}}{\text{unit investment cost } - b}. \tag{1}$$

When the capacity for the engineers to raise external funds per plant b is limited due to the short funding horizon, there is possible under-investment.

Our model displays a rich interaction between funding horizons and plant dynamics. The owner of plant has a distinct choice on a plan for future maintenance. Either he pays the costs needed to maintain, or even improve, productivity with a view to staying in production over the long haul – call this his *continuing strategy.* Or he curbs maintenance costs and allows productivity to deteriorate slowly, to some point when he decides to exit and liquidate plant as a generic building – call this his *stopping strategy.* This dichotomous decision – either planning to continue for the long haul, or planning to stop within a finite horizon – reveals an intriguing feature of equilibrium. For an open set of parameters, even though all plant starts off identical in productivity, their qualities diverge over time: some plant improves in productivity and other plant deteriorates and eventually shuts down.[7] We think all this may be a fruitful new vein for research into firm/plant dynamics, which should inform the study of how aggregate productivity evolves.

A question of particular concern to us is whether persistently low real interest rates can stifle aggregate investment and growth. The question is motivated by Japan, where the economy struggles to regain robust growth despite interest rates having been close to zero for over two decades. With this in mind, we model a small open economy where the world interest rate, R, is taken to be exogenous. What happens if the real interest rate R falls permanently?

The effects depend upon the impact on the numerator as well as on the denominator of the above equation (1). The net worth of engineers in the numerator rises as the value of their asset rises with a fall in R. In the denominator, the investment cost includes the price of building which tends to have a longer duration than the borrowing capacity. Then with a persistent fall in R, the borrowing capacity may fail to catch up with the rise of investment

[7] Griliches and Regev (1995) presents evidence that the productivity of many firms starts deteriorating before exiting, calling it the "shadow of death."

cost, pushing up the denominator.

There may be a parallel here with the housing market. When interest rates fall persistently, we often observe housing prices rising more than borrowing capacities. As a result, first-time buyers have a hard time getting onto the housing ladder. On the other hand, people who already own their houses enjoy a large increase in net worth and can afford to move into larger properties.[8] In the long run, the housing market may stagnate if not many new buyers enter into the market.

Along a time path following an unexpected fall in R, the rise in net worth initially dominates the rise in the required downpayment, leading to a temporary boom – especially if the liability to foreign creditors is not indexed. But the rise in the required downpayment can eventually overwhelm the rise in net worth. Overall, we demonstrate that domestic investment can *fall* with a fall in world interest rates, as can the growth rate in the long run.

We believe the dynamic path we uncover here – especially the disjunction between the initial rise in the value of total investment (including real estate) versus the subsequent fall in underlying productive investment in plant and human capital – may provide a sobering account of a number of property-fuelled booms sparked by lower interest rates. In particular, it may give a less rosy picture of, say, the Japanese property boom in the late 1980s, or the property boom in southern Europe following the introduction of the euro in the early 2000s, than more popular narratives based on asset price bubbles.

Lastly, in terms of policy intervention, we show there can be scope for an investment subsidy, financed by a plant-level payroll tax on engineers' maintenance services. This policy can improve welfare by implementing a form of additional group borrowing by the engineers, although the success of any such policy depends on exactly what the government is able to monitor.

The plan of the paper is as follows. In the next section, we lay out the model. Section 3 describes equilibrium in the part of the parameter space where no plant owner chooses to shut down his plant in finite time – what we call the Pure Equilibrium with No Stopping. Section 4 considers the complemetary part of the parameter space, where some plant owners choose to continue and others choose to stop – what we call the Mixed Equilibrium. Section 5 looks at the effects of a fall in the interest rate. Section 6 extends the model to allow for idiosyncratic uncertainty across plant. Finally, Section 7 considers policy intervention.

[8]Kiyotaki and Moore (1997).

2 Model

We consider a small open economy with an exogenous world real interest rate R. There is no aggregate uncertainty and, for the moment, we focus on steady state equilibrium (later, we will examine the effects of an unanticipated persistent drop in R). There is a homogeneous perishable consumption/investment good at each date $t = 0, 1, 2, \ldots$. We use this good as numeraire as we consider a non-monetary economy.

There is a $[0, 1]$ continuum of domestic agents, each maximizing utility of consumption c_t from the present to the infinite future:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right], \tag{2}$$

where $\beta \in (0, 1)$ is the utility discount factor and $\ln c$ is the natural logarithm of c. We assume that the exogenous world interest rate is nonnegative in net terms and lower than the subjective interest rate:

$$1 \leq R < \frac{1}{\beta}. \tag{3}$$

Each agent sometimes has an investment opportunity (being an entrepreneur or simply "engineer"), and sometimes not ("saver"). The transition probabilities of being an engineer conditional on being an engineer or a saver in the previous period are given by

$$\begin{aligned} \text{Prob}\,(\text{engineer at } t \mid \text{engineer at } t-1) &= \pi^E, \\ \text{Prob}\,(\text{engineer at } t \mid \text{saver at } t-1) &= \pi^S. \end{aligned}$$

We assume the arrival of an investment opportunity is persistent to a limited degree so that $0 \leq \pi^S \leq \pi^E < 1$.

At each date t, an engineer E can jointly produce plant and tools from goods and building: within the period, per unit of plant,

$$\left. \begin{array}{c} x \text{ goods} \\ 1 \text{ building} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{plant of productivity } 1 \\ 1 \text{ E-tool} \end{array} \right. . \tag{4}$$

The investment technology is constant returns to scale and scalable by any positive number i. Investment takes place after output has been produced, and plant and tools constructed are ready for use from date $t + 1$. Here we can think of tools as the engineer's human capital acquired through her learning by doing. As in Arrow (1962), the learning by doing is associated with gross investment instead of regular production.

Each tool (or human capital) is specific to the engineer ("E-tool") in that only she knows how to use it – unless she sells it to another engineer and teaches him. Because the engineer cannot usefully sell her tools to savers and her human capital is inalienable, she raises funds by selling all she can, the plant.

The plant owner has a constant returns to scale production technology. Match between plant and engineer is not specific - not specific even in present period, unlike in Introduction. At each date, the owner of one unit of plant of productivity z can hire any number $h \geq 0$ of tools (hiring each tool along with the engineer who knows how to use it) at a competitive rental price w ("wage") to produce goods and maintain plant productivity: within the period, per unit of plant,

$$\left.\begin{array}{c}\text{plant of productivity } z \\ h \text{ tools}\end{array}\right\} \rightarrow \left\{\begin{array}{c} y = az \text{ goods} \\ \lambda \text{ plant of productivity } z' = z^{\theta} h^{\eta} \\ \lambda h \text{ tools}\end{array}\right. . \quad (5)$$

$a > 0$ is the common productivity of all plant and z' is plant productivity after maintenance. $\lambda < 1$ reflects depreciation, by which a fraction $1 - \lambda$ of plant and tools are destroyed after use. The parameter θ is the share of present plant productivity and η is the share of engineers' tools (human capital) in maintaining plant productivity. We assume that $\theta, \eta > 0$, and $\theta + \eta \leq 1$. We can think of physical plant as tangible capital, productivity of plant as intangible capital, and both contributing to production at present and future.

Here, our sole departure from the Arrow-Debreu model is the assumption that engineers are unable to commit to work for less than the ex post competitive wage to the future plant owners (savers) at the time of investment. This form of constraint is sometimes called a non-exclusivity constraint.[9]

A brief word about interpretation is in order here. Although we call w the engineer's "wage", it is important to distinguish it from the simple wage of, say, an unskilled worker. The engineer's remuneration is like payment to a

[9] See, for example, Allen (1985), Townsend (1989), Cole and Kocherlakota (2001) and Attar, Mariotti and Salanie (2011).

skilled core employee who influences the firm's future productivity.[10] Notice that, unlike in a more fleshed-out macroeconomic framework, we assume a simple reduced-form production of output: proportional to the productivity of the plant. In Appendix A we show that this formulation is justified when output is a general decreasing returns to scale function of plant productivity and unskilled labor, where unskilled labor is hired by plant owners in a competitive market.

New buildings are supplied by foreigners. We assume foreign builders have an alternative use of building to produces a fixed amount of goods f per unit every period and the a building depreciate at the same rate as plant as

$$1 \text{ building} \rightarrow \left\{ \begin{array}{c} f \text{ goods} \\ \lambda \text{ building} \end{array} \right. .$$

Because foreigners are indifferent between supplying building to home engineers and using building themselves, the price building q is

$$q = \frac{f}{R} + \frac{\lambda f}{R^2} + \frac{\lambda^2 f}{R^3} + ... = \frac{f}{R - \lambda}.$$

Competitive foreign builders have enough capacity to satisfy the building demand of the domestic economy at their marginal cost q. We introduce foreign builders to ease the exposition.[11] Alternatively we can think engineers and plant owners rent building and pay the rent f per unit every period. Whether building is owned or rented by plant owners does not matter much the economic implication.[12]

[10] We takes the non-exclusivity constraint broadly: The trading history of engineers is private information, with which none can prevent engineers from selling their services and assets in the competitive market. Then, whether the remuneration is a spot payment or a claim to future revenue of the plant does not matter, because, to finance downpayment, an investing engineer can equally use a spot payment or the sale of a claim.

[11] Any difference between the building purchase price and the construction cost is the profit of foreign builders. If builders were domestic agents, we would need to take into account the impact of their income and wealth on the domestic economy – although we do not expect this would qualitatively change our findings.

[12] More generally, we can think plant owner needs to pay fixed cost f per period to operate the plant. If f is fixed cost for production, total fixed cost should be subtracted from GDP. Otherwise, it does not matter much whether f is opportunity cost of using building (as in text), rental price, or fixed cost of production (aside from the impact of a unexpected shock).

The plant owner always has an option to stop and convert the plant into generic building after production. The value of a unit of plant of productivity z at the end of the period is given by

$$V(z; w, R) = az + Max\left\{\lambda q,\ \max_{h\geq 0}\left[-wh + \frac{\lambda}{R}V\left(z^{\theta}h^{\eta}; w, R\right)\right]\right\}. \quad (6)$$

The first term in the right hand side (RHS) is the revenue from production. The value λq inside the braces is the value of stopping, while the second term is the value of continuing – minus wage cost and the capital value of the remaining λ units of plant with productivity $z' = z^{\theta}h^{\eta}$ in the next period.

Knowing that the return from maintaining plant productivity depends upon his future production and maintenance choices, the plant owner must devise a long-term plan: Either stop after a finite number of periods T, or continue forever $(T = \infty)$? For each $T = 0, 1, 2, \ldots$, define recursively the owner's value of a unit of plant of current productivity z stopping in T periods:

$$\begin{aligned} S^0(z; w, R) &= az + \lambda q, \\ S^1(z; w, R) &= az + \max_h\left[-wh + \frac{\lambda}{R}(az^{\theta}h^{\eta} + \lambda q)\right], \quad (7) \\ &\cdots \\ S^T(z; w, R) &= az + \max_h\left[-wh + \frac{\lambda}{R}S^{T-1}\left(z^{\theta}h^{\eta}; w, R\right)\right]. \quad (8) \end{aligned}$$

If the plant is shut down after production, the value $S^0(z; w, R)$ is $az + \lambda q$. If the plant owner shuts down in tomorrow after production, he hires tools today to balance the cost and benefit of maintaining plant productivity for production tomorrow. Generally, the owner's value of a unit of plant of current productivity z stopping in T periods, $S^T(z; w, R)$, equals the sum of present cash flow (revenue minus wage cost) and the present value of λ units of plant of productivity $z^{\theta}h^{\eta}$ stopping in $T - 1$ periods.

Now, for all value of z, the plant owner chooses the optimal stopping time so that

$$V(z; w, R) \equiv \sup_{T\geq 0} S^T(z; w, R). \quad (9)$$

Because new plant that she sells to a saver at the time of investment has productivity 1, the engineer raises funds, per unit of plant,

$$b = \frac{1}{R}V(1; w, R). \quad (10)$$

The value b can be thought of as the engineer's fund-raising capacity per unit of investment.

The required own-funds (downpayment) per unit of investment equals the gap between the investment cost and the fund-raising capacity,

$$x + q - b.$$

We assume that a new saver – an engineer yesterday who switched to being a saver today – can sell her tools (after use today) to an engineer, and teach him how to use them, at a competitive price $x + q - b$.

The budget constraint of an agent at date t who has h_t tools and d_t financial assets is

$$c_t + (x + q - b)i_t + \frac{d_{t+1}}{R} = wh_t + d_t,$$

where h_t is positive only if the agent was an engineer yesterday. Here, financial assets consist of the value of plant ownership as well as maturing one-period discount bonds. The discount bond is traded internationally at the interest rate R. Only if the agent is an engineer today, investment i_t can be positive and her tools tomorrow will be

$$h_{t+1} = \lambda h_t + i_t.$$

The budget constraint can be rewritten as

$$c_t + (x + q - b)h_{t+1} + \frac{d_{t+1}}{R} = [w + \lambda(x + q - b)]h_t + d_t = n_t,$$

where n_t is net worth – the sum of flow return (wage) and capital value (replacement cost or resale value) of tools, plus financial assets.

The rate of return for an engineer investing with maximal fund-raising is given by

$$R^E = \frac{w + \lambda(x + q - b)}{x + q - b}, \tag{11}$$

the ratio of total returns of one tool to the downpayment of investment. (Remember she sells plant to a saver at the time of investment and so does not receive the return on plant.) If the return on investment R^E exceeds the interest rate R, then, thanks to the logarithmic utility function, the engineer's consumption and investment are

$$c_t = (1 - \beta)n_t, \tag{12a}$$

$$(x + q - b)h_{t+1} = \beta n_t. \tag{12b}$$

And a saver's consumption and asset holdings are

$$c_t = (1-\beta)n_t, \tag{13a}$$

$$\frac{d_{t+1}}{R} = \beta n_t. \tag{13b}$$

Notice that individual consumption depends only on present net worth and not on whether the agent has an investment opportunity today. Because marginal utility is independent of whether or not there is an investment opportunity, there are no gains from insurance (such as the agent receives a bonus if she has an investment opportunity while pays a premium if not).

A steady state equilibrium of our small open economy is characterized by the wage w and the new plant price b, together with the quantity choices of savers/plant owners (c, d, h, z, y), engineers (c, h, i), and foreigners (who have net asset holdings D^*), such that the markets for goods, tools, plant, and bonds all clear.

3 Pure Equilibrium with No Stopping

We use a guess and verify method to characterize equilibrium. Suppose that in the steady state, no plant owner shuts down his plant until it depreciates exogenously. Then the value function (6) is the present value of net cash flow into the indefinite future:

$$V(z; w, R) = y_t - wh_t + \frac{\lambda}{R}(y_{t+1} - wh_{t+1}) + \left(\frac{\lambda}{R}\right)^2 (y_{t+2} - wh_{t+2}) + ...$$

An optimal sequence $\{h_t, z_{t+1}, h_{t+1}, z_{t+2},\ h_{t+2}, \ldots\}$ equates the discounted sum of marginal product to the wage (see Appendix B for the derivation):

$$w = \frac{\lambda}{R} a\eta \frac{z_{t+1}}{h_t} + \left(\frac{\lambda}{R}\right)^2 a\eta \frac{z_{t+1}}{h_t}\theta\frac{z_{t+2}}{z_{t+1}} + \left(\frac{\lambda}{R}\right)^3 a\eta \frac{z_{t+1}}{h_t}\theta\frac{z_{t+2}}{z_{t+1}}\theta\frac{z_{t+3}}{z_{t+2}} + \ \ldots \tag{14}$$

The first term in the RHS is the marginal impact of a date-t tool on output y_{t+1} through its impact on plant productivity z_{t+1}. The second term is the marginal impact of the date-t tool on y_{t+2} through its impact on z_{t+1} which impacts z_{t+2}. The third term is the marginal impact of the date-t tool on y_{t+3} through its impact on z_{t+1} which impacts z_{t+2} which in turn impacts z_{t+3}.

Multiplying through by h_t and simplifying, we get

$$wh_t = \frac{\lambda}{R}\eta y_{t+1} + \left(\frac{\lambda}{R}\right)^2 \eta\theta y_{t+2} + \left(\frac{\lambda}{R}\right)^3 \eta\theta^2 y_{t+3} + \dots \tag{15}$$

The present wage bill for engineers equals the present discounted value of a fraction η of tomorrow's output, plus a fraction $\eta\theta$ of output two periods later, plus a fraction $\eta\theta^2$ of output three periods later, etc.

An engineer raises funds by selling new plant at price

$$\begin{aligned} b &= \frac{1}{R}V(1; w, R) \\ &= \frac{1}{R}a + \frac{\lambda}{R^2}y_{t+1}(1-\eta) + \frac{\lambda^2}{R^3}y_{t+2}(1-\eta-\eta\theta) + \ \dots \end{aligned} \tag{16}$$

All plant starts with productivity $z = 1$. Moreover, investment generates an equal number of plant and tools, which have the same technological depreciation rate $1 - \lambda$. If no plant is stopped, the ratio of tools to plant stays one-to-one. Then because

$$z' = z^\theta h^\eta = 1 \text{ when } z = h = 1,$$

all plant is maintained at initial productivity $z = 1$ until the exogenous death of plant through depreciation. Output per unit of plant is unchanged from the initial level:

$$y_{t+1} = y_{t+2} = y_{t+3} = \dots = a.$$

The engineer's fund-raising capacity b becomes

$$b = \frac{1}{R}a \ + \frac{\lambda}{R^2}a(1-\eta) + \frac{\lambda^2}{R^3}a(1-\eta-\eta\theta) + \dots \tag{17}$$

Notice how the plant owner's share declines in more distant future output: 1, $1-\eta$, $1-\eta-\eta\theta, \dots$

Figure 1 shows how future output is split between the plant owner (who provides the external finance) and the engineers (who maintain the plant). The horizontal axis is time horizon: t measures how far distant future is from present. The vertical axis is output per unit of surviving plant, and the shares of future output for the plant owner and engineers as functions of time horizon. The parameters and equilibrium wage rate w shown in the figure. In pure equilibrium with no stopping, output is $y_t = a$. The downward sloping

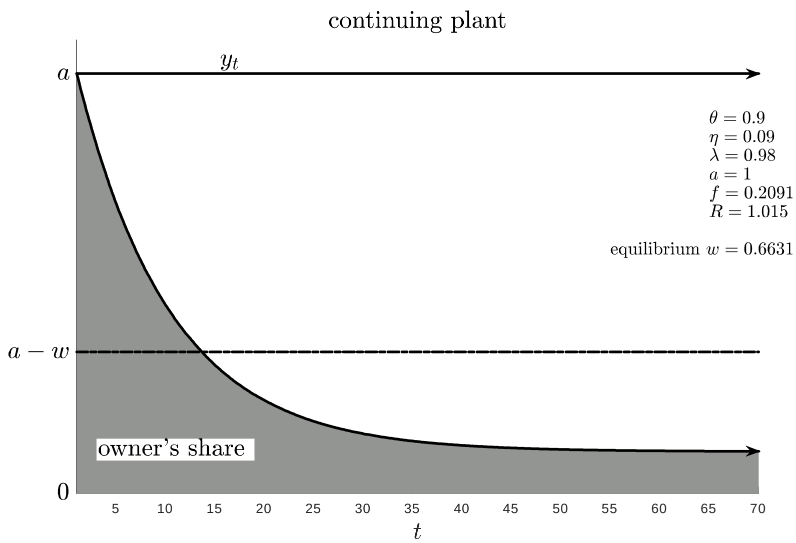

Figure 1: Horizons of owner's contribution to firm's revenues.

curve is the share of future output the plant owner obtains. Think of this as his rightful reward for his "contribution" to those revenues, stemming from the initial productivity $z = 1$ of the plant that he paid to own.

Through investment the engineer acquires tools (human capital) which is just enough to maintain the productivity of plant at the initial productivity level in Pure Equilibrium with No Stopping. Thus her payoff from unit investment equals the present value of the engineers' wage as

$$\begin{aligned} & \frac{1}{R}w + \frac{\lambda}{R^2}w + \frac{\lambda^2}{R^3}w + \dots \\ = \quad & 0 + \frac{\lambda}{R^2}a\eta + \frac{\lambda^2}{R^3}a(\eta + \eta\theta) + \frac{\lambda^3}{R^4}a(\eta + \eta\theta + \eta\theta^2) + \dots \end{aligned} \tag{18}$$

we see that, correspondingly, the engineers' share rises in more distant future output: $0, \eta, \eta+\eta\theta, \eta+\eta\theta+\eta\theta^2, \dots$ The gap between output and the owner's share in Figure 1 is the engineers' share of future output. Intuitively, as the cumulative contribution of engineers' human capital to plant productivity rises with horizon, the effects of the plant's initial productivity – essentially what a saver gets when he buys a new plant – tails off. Figure 1 illustrates why the plant owner's share (investing engineer's pledgeable return) is largely near-term, and she raises funds against near future revenue.[13]

Take a special case of constant returns to scale maintenance technology $\eta + \theta = 1$ and no depreciation $\lambda = 1$. The value function of plant with no stopping is

$$V(z) = az + \underset{h}{Max}\left[-wh + \frac{1}{R}V(z^{\theta}h^{1-\theta})\right].$$

The first order condition for h implies

$$w = \frac{1}{R}(1-\theta)V'(z')\frac{z'}{h} = \frac{1}{R}(1-\theta)V'(z')\left(\frac{z}{h}\right)^{\theta}. \tag{19}$$

We guess the value function to be proportional to z as

$$\begin{aligned} V(z) &= \upsilon z = az + \frac{\theta}{R}\upsilon z' \\ &= az + \frac{\theta}{R}V(z'). \end{aligned} \tag{20}$$

Because h and z' are proportional to z from (19) and $V'(z) = \upsilon$, we verify the guess.

This expression (20) is identical to the bilateral bargaining between plant owner and engineer if their bargaining powers are θ and $1-\theta$. In bilateral bargaining, because the plant owner needs the engineer every period, he only obtains $\theta < 1$ fraction of the continuation value of the plant at the end of this period. In our baseline model, because the wage bill of engineers equals a fraction $1-\theta$ of the next period value $V'(z')z' = V(z')$ under constant returns to scale maintenance technology from (19) , the plant owner captures only θ fraction of the continuation value as in (20).[14]

[13] Notice that Figure 1 shows which shares of future output belongs to the plant owner vs. engineers as a function of time horizon – how distant is the future from now. Because wage reflects contribution to future output and past wage payment is sunk cost, the realized division of output does not change with the age of plant in Pure Equilibrium with No Stopping: $a - w$ to the plant owner and w to engineers.

Thus the implication our funding horizon model are very different from dynamic contract literature in which the principal chooses the agents' wage to be back-loaded in order to induce her effort, when the earning depends upon the agent's unobservable effort and idiosyncratic temporary shocks.

[14] These conditions resemble "Hosio's condition" in the labor matching literature.

In particular, in Pure Equilibrium with No Stopping, we have $z = h = 1 = z'$ and

$$b = \frac{1}{R}V(1) = \frac{1}{R}v = \frac{a}{R - \theta}.$$

In both bilateral bargaining and competitive market cases, the saver is willing to provide fund to the investing engineer at the time of investment only against near-term revenue. An advantage of using a competitive framework with non-exclusivity constraint is that we can analyze dynamics of economy without specifying details of bilateral bargaining when each agent may or may not have an investment opportunities at each future date.

Returning to the general baseline model, we can aggregate across engineers and savers to obtain aggregate tool holdings H_{t+1}, financial asset holdings D_{t+1}/R, consumption C_t, and net worth of engineers and savers $\left(N_t^E \text{ and } N_t^S\right)$:

$$\begin{aligned}
(x + q - b)H_{t+1} &= \beta N_t^E, && (21\text{a}) \\
\frac{D_{t+1}}{R} &= \beta N_t^S, && (21\text{b}) \\
C_t &= (1 - \beta)\left(N_t^E + N_t^S\right), && (21\text{c}) \\
N_t^E &= \pi^E \left[w + \lambda(x + q - b)\right] H_t + \pi^S D_t, && (21\text{d}) \\
N_t^S &= \left(1 - \pi^E\right) \left[w + \lambda(x + q - b)\right] H_t + (1 - \pi^S) D_t. && (21\text{e})
\end{aligned}$$

Equation $(21a)$ says the aggregate capital value of tools equals the aggregate net worth of engineers after subtracting their consumption, and equation $(21b)$ says the aggregate value of financial assets equals the aggregate net worth of savers after consumption. In equation $(21c)$, aggregate consumption equals a fraction $1 - \beta$ of the aggregate net worth of domestic residents. The aggregate net worth of engineers is the sum of the net worth of continuing and new engineers in equation $(21d)$, and the aggregate net worth of savers is sum of the net worth of new and continuing savers in equation $(21e)$.

The economy exhibits endogenous growth G: along a steady state path, such that

$$\frac{H_{t+1}}{H_t} = \frac{D_{t+1}}{D_t} = \frac{C_{t+1}}{C_t} = G,$$

$$\begin{aligned}
GN_t^E &= N_{t+1}^E = \pi^E R^E \beta N_t^E + \pi^S R \beta N_t^S, \\
GN_t^S &= N_{t+1}^S = (1 - \pi^E) R^E \beta N_t^E + (1 - \pi^S) R \beta N_t^S.
\end{aligned}$$

Substituting out $\frac{N_t^s}{N_t^E}$, we find that G solves

$$G = \pi^E R^E \beta + \pi^S R\beta \frac{(1-\pi^E) R^E \beta}{G - (1-\pi^S) R\beta}. \tag{22}$$

The growth rate depends on the rates of return for engineers and savers as well as on the wealth distribution between them.

Now, under certain conditions, we can verify our initial guess that no plant owner stops in the steady state:[15]

Proposition 1: If the opportunity cost for using building for production is smaller than some critical value $f^{critical}$, then there is a steady state equilibrium in which

(a) no plant owner stops;

(b) the aggregate ratio of tools to plant stays one-to-one: $h = 1$;

(c) all plant is maintained at the initial productivity level: $z = z^* = 1$;

(d) All plant has output $y = a$.

We call this a **Pure Equilibrium with No Stopping**, that exists when the model's parameters lie in the **P-Region**. In Appendix B, we derive a sufficient (but not necessary) condition for the existence of a pure equilibrium with no stopping:

$$f < a\frac{R(1-\theta-\eta)}{\lambda(1-\theta)}\left[1 - \frac{R-\lambda}{R}\left(\frac{R-\lambda\theta}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\right]. \tag{23}$$

4 Mixed Equilibrium

What happens if the condition for the pure equilibrium with no stopping is violated, i.e. the opportunity cost is higher than the critical level $f^{critical}$ in Proposition 1? It turns out there is a clear dichotomy for the plant owner between continuing forever and stopping after a finite number of periods (for a given wage and interest rate):

Lemma:

(a) If the current plant productivity z is below some cutoff value, $z^\dagger$, it is

[15] All proofs and details of derivations are in Appendix B. Proposition 3(b) is demonstrated numerically.

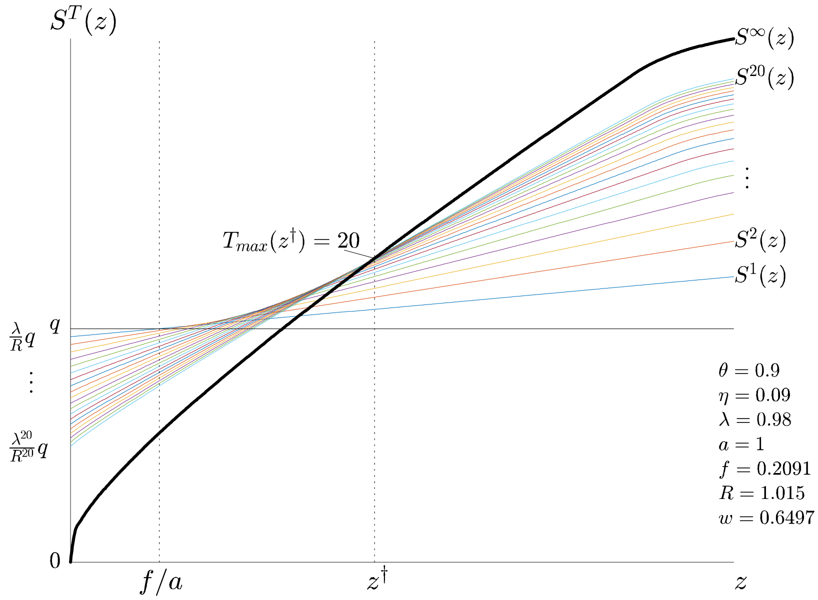

Figure 2: Value functions and stopping horizons.

optimal for the plant owner to stop after, say, $T_{\max}(z) < \infty$ periods.
(b) If z is above $z^\dagger$, it is optimal to continue forever.
(c) The cutoff value $z^\dagger$ increases with the opportunity cost f. It is also a function of the wage rate and the interest rate.

In Figure 2, we plot the per-unit plant value, as a function of the current productivity z, for a given wage w and interest rate R, and for different stopping horizons T. The function $S^\infty(z; w, R)$ is the value when the plant owner chooses to maintain production forever. The upper envelope of all these functions is the value function of plant $V(z; w, R)$ with an optimal choice of stopping (including non-stopping). If z is very low, then it is optimal for the owner to shut down the plant immediately after present production. Thus, if z is not very low but lower than $z^\dagger$, the owner will shut down plant not immediately but in a finite horizon, where the horizon is an increasing function of z. At $z = z^\dagger$, the plant owner is indifferent between continuing forever and shutting down in a finite time (for this numerical example, in 20 periods). If z is higher than $z^\dagger$, the plant owner will continue forever – that is, until the plant dies exogenously.

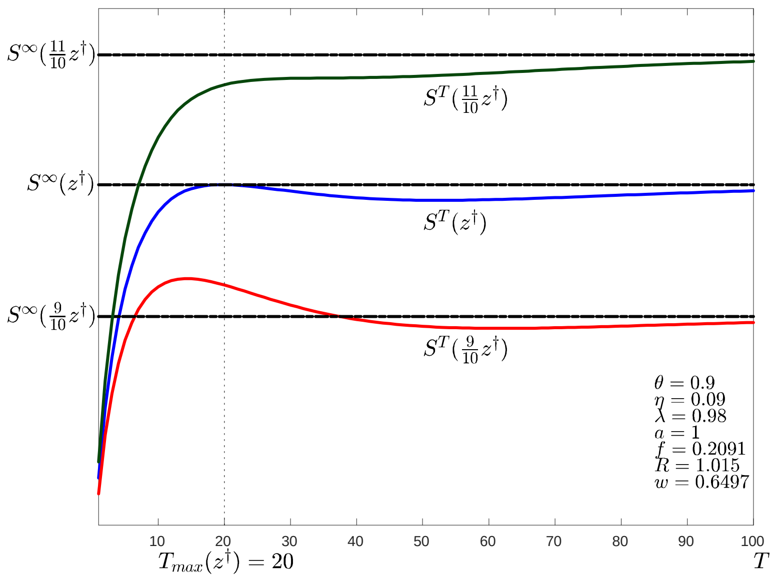

Figure 3: Value functions $S^T(z)$ near threshold $z^{\dagger}$ where plant owner is indifferent between stopping in a finite horizon or continuing forever.

In Figure 3, we plot $S^T(z; w, R)$ as a function of horizon T for three different levels of plant productivity, $z = 0.9z^\dagger$, $z^\dagger$, and $1.1z^\dagger$. If plant productivity is relatively low, at $z = 0.9z^\dagger$, then the value reaches a maximum with finite horizon: for our example, around $T = 15$ so that the owner will shut down in 15 periods. If plant productivity is exactly equal to $z^\dagger$, then the plant owner is indifferent between shutting down in 20 periods ($T = 20$) and continuing forever ($T = \infty$). If plant productivity is relatively high, at $z = 1.1z^\dagger$, then the owner finds that $S^\infty(z; w, R) > S^T(z; w, R)$ for any finite T so that he will continue forever.

In general equilibrium, the wage rate w and the value of $z^\dagger$ are endogenous. The aggregate dynamics of net worth, tools, financial asset holdings and consumption are still described by equations $(21a)-(22)$ but, in contrast to Proposition 1, there is now stopping:

Proposition 2: If the opportunity cost for operating a unit of plant f is larger than a critical value $f^{critical}$ from Proposition 1, then there is an equilibrium in which**:**

(a) Plant owners are initially indifferent between stopping in finite time T and continuing forever: $z^\dagger = 1$; in particular,

(i) if the initial output is larger than the opportunity cost, $a > f$, then plant owners are initially indifferent between stopping in finite time $T \geq 1$ and continuing forever, whereas

(ii) if the initial output is smaller than the opportunity cost ($a < f$), then plant owners are initially indifferent between stopping immediately ($T = 0$) and continuing forever;

(b) The aggregate ratio of tools-to-plant is larger than one-to-one for continuing plant: $h > 1$;

(c) With decreasing returns to scale, $\theta + \eta < 1$, the productivity of continuing plant increases over time, converging to some $z^* \in (1, \infty)$; whereas with constant returns to scale, $\theta + \eta = 1$, the productivity of continuing plant grows at some constant rate $g > 1$;

(d) If $f \in (f^{critical}, a)$, then the productivity of stopping plant decreases over time;

(e) There is no equilibrium where all plant stops in finite time.

We call this a **Mixed Equilibrium**, that exists when the model's parameters lie in the **M-Region** (the complement of the P-Region). Within this region, the initial productivity is exactly equal to the critical productivity $z^\dagger$

for shutting down, so that some plant is stopped and some continues forever (modulo depreciation). Because the owners of stopping plant do not hire many tools, the aggregate ratio of tools to plant is larger than one-to-one for continuing plant: $h > 1$. With an abundant supply of tools per plant, continuing plant keeps improving in productivity. If the maintenance technology has decreasing returns to scale, $\theta+\eta < 1$, the productivity of continuing plant converges to some finite steady state level z^*. If the maintenance technology has constant returns to scale, $\theta + \eta = 1$, the productivity of continuing plant grows at some rate $g > 1$. Therefore, even though all plant is homogeneous when new, some plant improves in productivity while the rest fails to maintain productivity and eventually exits (or immediately exits if $a < f$). That is, firms evolve heterogeneously in their productivity and output even though they start off homogenous and face no idiosyncratic shocks.[16],[17]

If all plant were to stop in finite time, the market for tools (engineers) would be in excess supply: because of exit the quantity of active plant would be smaller than tools, plus there would be little demand for tools by plant owners who are planning to stop, so the equilibrium wage rate for tools would fall to the point where at least some plant owners switch strategy and continue forever.

5 Effects of Falling Interest Rate

Figure 4 adds to Figure 1 the opportunity cost of using operating plant instead of liquidating in Pure Equilibrium with No Stopping. The grey and red heights are the plant owner's share of output net of the opportunity cost f. In the near horizon his net share is positive (the grey area), as might be expected. But in the far horizon his net share has switched to become negative (the red area).

[16] This is different from the standard approach taken by Jovanovic (1981) and Hopenhayn (1992) in which initial heterogeneity and/or subsequent heterogeneity (induced by idiosyncratic shocks) are essential to firm dynamics. Even allowing for idiosyncratic shocks (see Section 5), our approach may provide a different perspective on firm dynamics. Our model is more closely related to, for example, Atkeson and Burnstein (2010), Clementi and Palazzo (2016), Ericson and Pakes (1995), Klette and Kortum (2004), and Rossi-Hansberg and Wright (2007), all of which stress the interaction between heterogeneity, idiosyncratic shocks, and investment.

[17] Allowing for initial heterogeneity would purify this mixed-strategy equilibrium so that

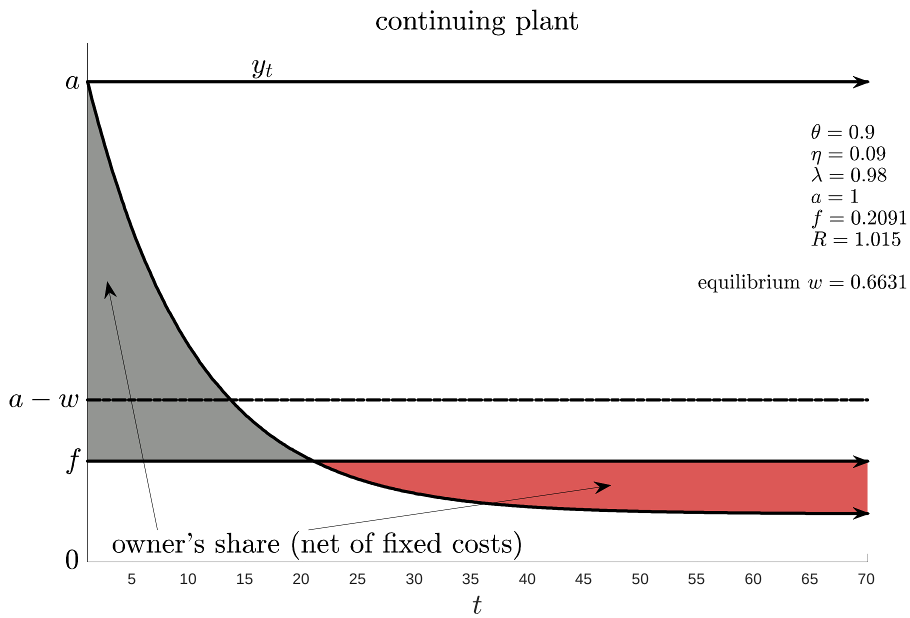

Figure 4: Horizons of owner's contribution to firm's revenues.

This begs the question: Why doesn't the plant owner shut down at this point and liquidate the plant into generic building to recover the building value? The reason is that, while the present wage bill equals the present value of engineers' current contribution to future revenues, *past* wage bills are sunk costs for the plant owner. As long as the present value of *future* profit exceeds that of opportunity cost, $PV\left(y_{t+\tau} - w_{t+\tau}\right) > PV\left(f\right) = q$, the owner chooses to continue with maintenance and production.

Because terms in the more distant future are more sensitive to a permanent change of interest rate, a fall in R may not expand the engineer's fund-raising capacity as much as the building price. In particular, when $\theta + \eta = 1$, we can solve (17):

$$b = \frac{a}{R - \lambda\theta}. \tag{24}$$

Notice that the plant owner's share of gross output decreases with the horizon by factor $\lambda\theta$, because the owner in effect pays to engineers an increasing

plant owners would, more realistically, follow pure strategies. See Aumann et al. (1983).

fraction of more distant future output for their maintenance services. In contrast, the engineer's investment cost per unit equals

$$x + q = x + \frac{f}{R - \lambda}$$

which is the sum of goods and building cost per unit. Since $\lambda > \lambda\theta$, building has a longer horizon than the owner's share of gross output: a fall in R may not expand the present value of the plant owner's share of gross revenues (engineer's pledgeable returns) in (24) as much as the unit investment cost. This can increase the downpayment $x + q - b$ which the engineer has to pay from her net worth.

Proposition 3 (Pure Equilibrium with No Stopping):

(a) For an open subset of the P-Region, in particular for R and λ not too far from unity, there is a pure equilibrium with no stopping such that an unexpected permanent drop in the interest rate R leads to a lower steady state growth.[18]

(b) Immediately following the drop in R, all agents (engineers and savers) can be strictly worse off.

To understand why a fall in R can stifle investment and growth, consider the effect on the equation for gross investment:

$$\text{gross investment } (H_{t+1}) \downarrow \quad =$$

[18] A sufficient condition for the steady state growth rate to fall with a permanent drop in interest rate in pure equilibrium with no stopping is that $\pi^S = 0$ and

$$f > a\frac{R - \lambda(\theta + \eta)}{R - \lambda\theta} - a\frac{G - \beta R\pi^E}{G - \beta\lambda\pi^E}\frac{\lambda\eta(R - \lambda)}{(R - \lambda\theta)^2}.$$

This inequality and a sufficient condition for the existence of Pure Equilibrium with No Stopping (23) are mutually consistent if R and λ are not too far from unity.

If $\pi^S > 0$, then a sufficient condition for the growth rate to fall with permanent fall in the interest rate is that

$$\lambda(1 - \theta) f > (R - \lambda)^2 x + \lambda(1 - \theta - \eta)a.$$

This condition guarantees that the rate of return for an investing engineer is an increasing function of the interest rate. See Appendix B. Because – unlike the previous sufficient condition – this condition involves x, it cannot be readily juxtaposed with (23).

$$\underset{\text{saving rate}}{\beta} \times \frac{\text{net worth of engineers } \left(N_t^E\right) \uparrow}{\text{investment cost } \ (x+q) \uparrow\uparrow \ - \text{borrowing capacity} \, (b) \uparrow}.$$

Although engineers' net worth increases with a fall in R (primarily through a rise their wage w), a decrease in their borrowing capacity may have a larger negative effect on investment, and therefore on growth. Much of the macro finance literature (including Bernanke and Gertler (1989) and Kiyotaki and Moore (1997)) has emphasized effects on net worth in the numerator. Here we are focussing on the effect on borrowing capacity in the denominator.

In terms of welfare, a fall in R can make all domestic agents (engineers and savers) strictly worse off. It is not surprising that savers may be worse off with a lower rate of return on financial assets. The reason why engineers may be worse off is that their leveraged rate of return

$$R^E = \frac{w + \lambda(x+q-b)}{x+q-b}$$

can fall though increase of downpayment $x + q - b$. Appendix B derives the welfare of engineers and savers immediately after an unanticipated and permanent fall in the interest rate, taking into account the stochastic arrival of future investment opportunities.

Suppose the economy was in steady state at date $t-1$, for a presumed constant interest rate. Unexpectedly at date t, the interest rate falls permanently. If the parameters satisfy the condition of Proposition 3(a), then the long-run growth rate falls. Figure 5 shows the movement of the aggregate values of investment, consumption, output and foreign debt holdings, when the real interest rate unexpectedly falls from 2.5% to 1.5% permanently at date 5.

Initially, the measured value I_t^m of total investment increases because buildings are more expensive and the new engineers have greater net worth due to capital gains on plant they hold from the previous period. Consumption increases too, with the greater net worth. Because domestic absorption (investment and consumption) expands more than output, foreign debt rises rapidly during the transition. Despite the boom, the growth rate of plant and human capital eventually falls. As the boom fades, the slower growth of productive capacity, exacerbated by a larger foreign debt-to-income ratio, causes secular stagnation.[19]

[19] Here we examined the transition from a high interest-rate steady state to a low interest-

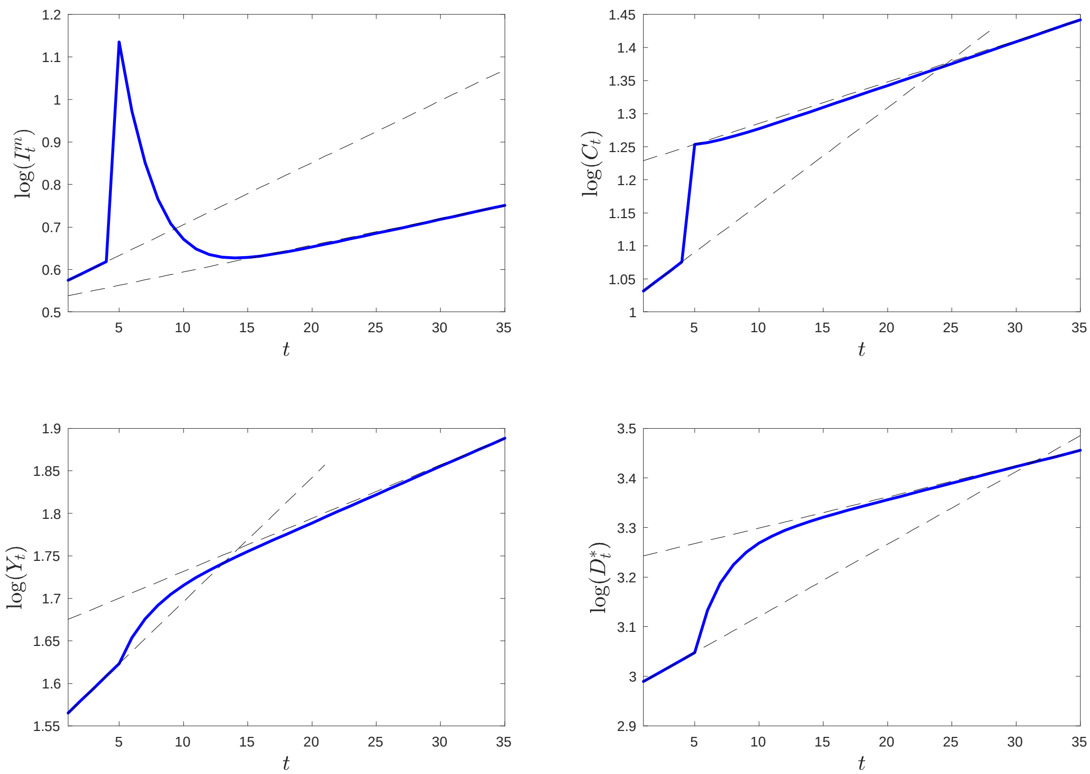

Figure 5: Impluse Response to Permanent Fall in Interest Rate

It has been observed that during the credit and asset price booms in Japan in the late 1980s and in southern Europe in the early 2000s, the aggregate values of credit and assets grew faster than productive capacity. (See Hoshi and Ito (2020) and Gopinath, Kalemi-Ozcan, Karabarbounis and Villegas-Sanchez (2017)). In the macro-finance literature, many authors have observed that credit booms associated with asset price booms are often followed by financial crises. (See for example, Reinhart and Rogoff (2008), Schularick and Taylor (2012) and Jorda, Schularick and Taylor (2018).) These authors consider such booms as being associated with excessive expansion of credit and assets values. Our model provides a different perspective, even though we do not deny the possibility of excessive assets values.

From this perspective, a persistently lower real interest rate leads to an initial credit and asset value boom, but stagnation in the long run, not because the boom was excessive, but because the underlying growth rate of the productive capacity declined.[20]

For the Mixed Equilibrium we have limited analytical results, and derive our findings by numerical simulations:

Proposition 4 (Mixed Equilibrium)

A lower interest rate R can be associated with a lower steady state growth rate G.

In Figure 6, we illustrate how nine endogenous variables depend on the interest rate R in the range between 1 and 1.03 (between 0 and 3% net) in

rate one. The result is very different if the interest rate fluctuates deterministically in medium run, say 5 year low followed by 5 year high. Then the borrowing capacity is likely to expand more than the building price in proportion. In addition the net worth of engineers increases. Hence, aggregate investment, external funds and output all tend to expand with a fall in the interest rate in the medium run. Therefore, a permanent fall in interest rate can stifle growth in the long run, even though a fall in interest rate in the medium run tends to be expansionary - as commonly observed.

The sequence of events described in Figure 5 may correspond better to southern European countries in the early 2000s than to Japan in late 1980s, insofar as the fall in their interest rate was fast and considered to be permanent.

[20] Another, complementary, perspective to ours is that credit and asset price booms associated with lower interest rates tend to lead to greater misallocation of capital when the domestic financial system is underdeveloped. See, for example, Aoki, Benigno and Kiyotaki (2007), Reis (2013), Gopinath, Kalemi-Ozcan, Karabarbounis and Villegas-Sanchez (2017), and Asriyan, Martin, Vanasco and Van der Ghote (2020).

steady state equilibrium. We choose the parameters so that the economy is in the pure no-stopping region (P-Region) for $R \in [1.015, 1.03]$ and in the mixed equilibrium region (M-Region) for $R \in [1, 1.015)$.

θ	share of past productivity in maintenance	0.9
η	share of engineer in maintenance	0.09
λ	one minus depreciation rate	0.98
a	productivity	1
f	opportunity cost	0.2091
x	investment cost per plant	6.127
β	utility discount factor	0.92
π^E	probability of staying to be engineer	0.7
π^S	probability of saver to become engineer	0.1

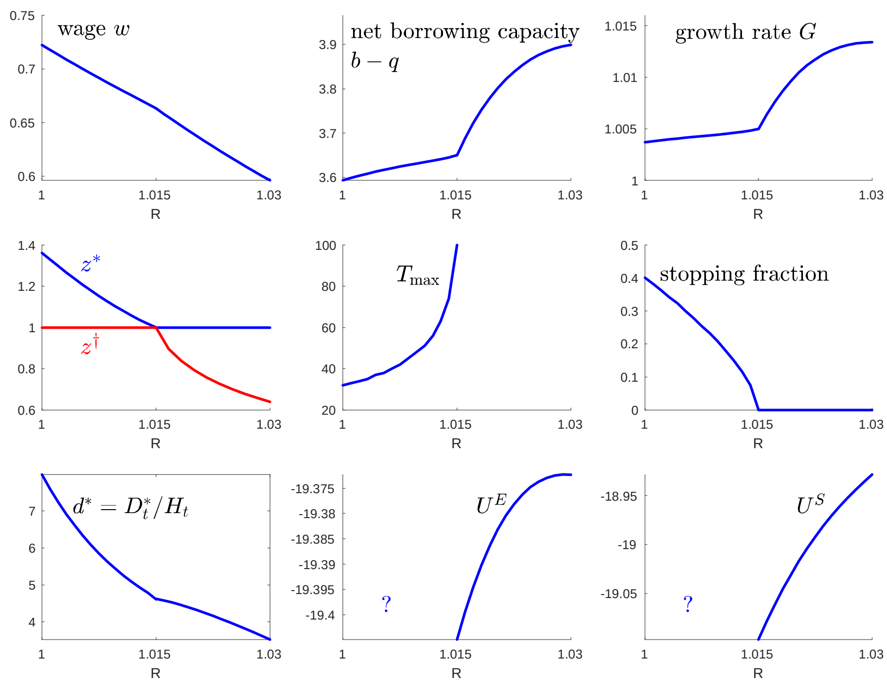

Figure 6: Lower real rate, credit horizons and stagnation.

In the top-left panel of Figure 6, the wage rate is a decreasing function of the interest rate because an engineer's contribution to future output through maintenance work is forward looking. In the top-middle panel, the difference between engineer's borrowing capacity and building price increases with the interest rate because the plant owner's share of output has a shorter duration than building. Notice that this effect is smaller in the M-Region, with an endogenous adjustment of the fraction of stopping plant (extensive margin) and of the stopping time (intensive margin). In the top-right panel, the economy's growth rate is an increasing function of interest rate, albeit that the sensitivity is weaker in the M-Region.

In the middle-left panel, the asymptotic plant productivity z^*equals 1 in the P-Region and is a decreasing function of R in the M-Region. The threshold plant productivity for continuing and stopping $z^\dagger$ equals 1 (initial productivity) in the M-Region (consistent with plant owners being indifferent between stopping and continuing) and is a decreasing function of R in the P-Region (consistent with plant owners gaining more indirectly from the lower wage rate than hurting directly from the higher interest rate). In the middle-middle panel, the number of periods before stopping ($T^{\max}$) is finite and is an increasing function of R for those who choose to stop in the M-Region. In the P-Region, no-one stops and $T^{\max} = \infty$. In the middle-right panel, the fraction of stopping plant is zero in the P-Region and is a decreasing function of R in the M-Region.

In the bottom-left panel, we see that the net financial asset holdings of foreigners is positive, i.e., domestic residents borrow from foreigners in net terms in the steady state. This is consistent with the home economy being growing and the foreign interest rate being lower than the subjective interest rate. With a lower interest rate, domestic residents borrow more from foreigners. In the bottom-middle panel, the welfare of a representative engineer (who holds the average net worth of engineers) is an increasing function of R in the P-Region, i.e., welfare is lower with lower R. In our example, the welfare of a representative engineer with $R = 1.015$ is lower than that of $R = 1.03$ by the equivalent of a 0.12% permanent fall in consumption. We do not have comparable results for the M-Region, because one cannot define simply what is meant by a representative engineer. In the bottom-right panel, the welfare of savers is an increasing function of R in the P-Region. The effect on savers is larger: their welfare with $R = 1.015$ is lower compared to $R = 1.03$ by the equivalent of a 1.2% permanent fall in consumption.

6 Extension: idiosyncratic uncertainty

In the model thus far, even though plant produces output deterministically, we find that equilibrium plant dynamics emerge in the mixed equilibrium where some plant owners hire insufficient engineers to maintain plant productivity and slowly exit. In this section, we further connect our theory to the literature on plant dynamics by introducing idiosyncratic shocks to plant productivity. We study how these shocks affect plant owners' decisions on maintenance and exit.

Let us modify the production technology (5) to:

$$\left.\begin{array}{c}\text{plant of productivity } z\\ h \text{ tools}\end{array}\right\} \rightarrow \left\{\begin{array}{c} y = az \text{ goods}\\ \lambda \text{ plant of productivity } z' = \epsilon z^{\theta} h^{\eta}\\ \lambda h \text{ tools}\end{array}\right. ,$$

where ϵ is an idiosyncratic productivity shock, i.i.d. across plant and over time. It follows a lognormal distribution whose mean is normalized to one:

$$\log \epsilon \sim N(-\frac{\sigma^2}{2}, \sigma^2).$$

The value of a unit of plant of productivity z at the end of a period is

$$V(z) = az + Max\left\{q,\ \max_h[-wh + \frac{\lambda}{R}EV(\epsilon z^{\theta} h^{\eta})]\right\}. \tag{25}$$

Compared with the plant value without productivity uncertainty, the only difference is that the continuation value of the firm is subject to the idiosyncratic shock, ϵ.

To illustrate the effect of the idiosyncratic shock, we continue with the numerical example in the previous sections ($\theta = 0.9$, $\eta = 0.09$, $\lambda = 0.98$, $a = 1$, $f = 0.2091$, $R = 1.015$, and $w = 0.6497$).

When the productivity shock has a small variance, the owner's productivity maintenance decision is similar to that in a deterministic environment. Figure 7 illustrates the maintenance decision, h, and the expected productivity in the following period, z', when idiosyncratic shocks have a small dispersion, $\sigma = 0.0001$. In this case, there still exists a dichotomy between those plants that the owners intend to exit and those that the owners intend to continue. If current plant productivity z is below a critical value, $z^{\dagger}$, the plant owner does not hire much maintenance service and most likely exits in

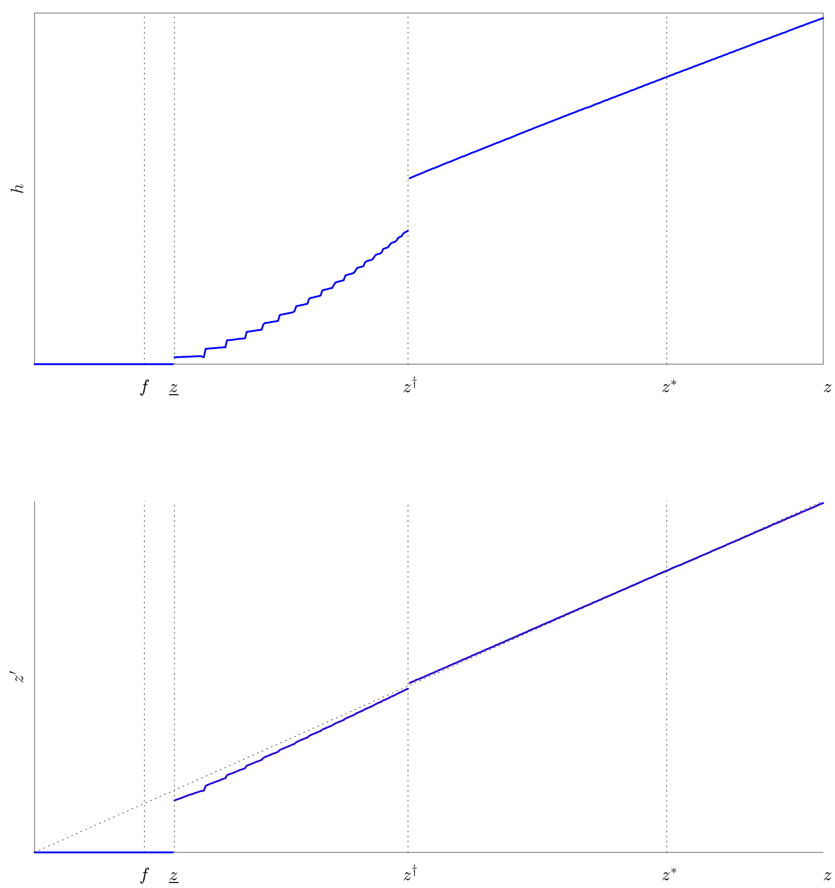

Figure 7: Productivity maintenance with small idiosyncratic risk, $\sigma = 0.0001$.

random finite number of periods but not immediately. If z is above $z^\dagger$, the plant owner hires distinctively larger engineering service and continues operating until the plant dies exogenously, unless extremely unlucky idiosyncratic shocks bring down the plant productivity below $z^\dagger$.

At $z = z^\dagger$, the plant owner has two distinct optimal levels of maintenance: his expected value from maintenance has twin peaks. If he chooses the slow exit strategy, he saves some maintenance costs but receives less profit from future production. If he chooses to continue operating the plant for the long haul, he pays more to maintain the plant and in return receives more profit from future production. The maintenance input h and expected productivity z' increase discontinuously as current productivity z moves up across the critical value $z^\dagger$, as the plant owner finds it optimal to operate the plant for the long haul.[21]

Figure 8 illustrates the plant owner's maintenance decision, h, and productivity distribution in the following period, z', when the idiosyncratic shocks are large, $\sigma = 0.02$. In the figure on realized productivity, z', the blue curve represents the expected productivity in the following period. The red curves represent the realized productivity that are three standard deviations above or below the expected value. With large productivity shocks, the dichotomy between exiting and continuing is blurred: the plant owner's maintenance input is a continuous function of the current plant productivity z. This is because even when the plant owner would like to improve productivity, a large negative idiosyncratic shock may still lead to a low productivity. This smooths the plant owner's expected payoff from maintenance and makes it single-peaked.

7 Policy

When the competitive equilibrium is not efficient, it is natural to ask whether the government could improve welfare through taxes and subsidies. The sole departure from the Arrow-Debreu model in our framework is the non-exclusivity constraint: a saver who buys plant from an engineer (the creditor who lends to the engineer against the plant) cannot prevent this engineer

[21] If we allow for heterogeneity of initial productivity, there is no mass of plant which have productivity exactly equal to $z^\dagger$.

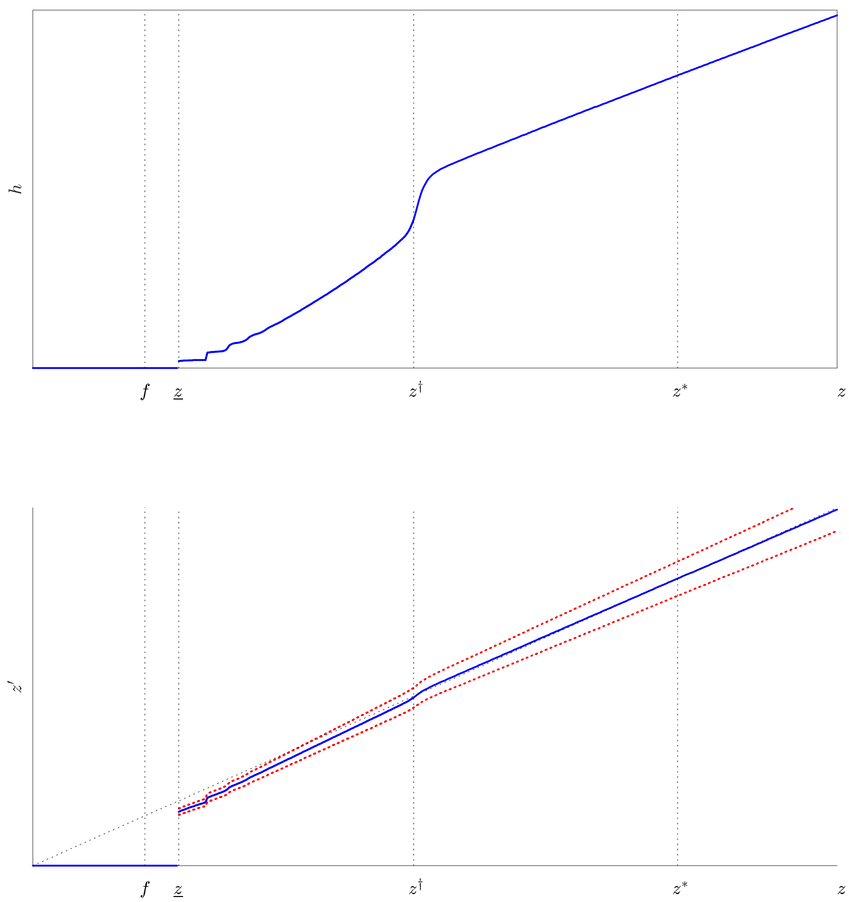

Figure 8: Productivity maintenance with large idiosyncratic risk, $\sigma = 0.02$.

from working for another plant in future. In effect, we are supposing it is impossible to keep track of each engineer's trading history.

However, because the plant is easy to locate, it may be possible for the government to keep track of how much the plant owner buys the maintenance services of engineers – even though government does not know the identity of engineers. Suppose government can tax the payroll for engineers of each plant owner at rate τ, and use the tax revenue to subsidize engineers by s per unit of investment. We ignore idiosyncratic shocks to the realized productivity after maintenance and restrict our attention to the steady state of a Pure Equilibrium with No Stopping (the parameters lie in Region P). We assume the government's budget is balanced:

$$\tau w^o H = sI = s(G - \lambda)H.$$

w^o is wage rate for plant owners, $\tau w^o H$ is the payroll tax revenue, and $sI = s(G - \lambda)H$ is the investment subsidy.

Because plant owners equate the marginal contribution of engineers' expertise to the wage rate (including the payroll tax), we have

$$w^o = \frac{\eta\lambda}{R - \lambda\theta} a, \tag{26}$$

using (14) with $h_t = z_t = 1$ in the steady state. Notice that the payroll tax does not affect the wage cost to the plant owner, but reduces the wage rate $w = (1 - \tau)w^o$ for engineers. Together, we get

$$s = \tau \frac{w^o}{G - \lambda}. \tag{27}$$

The price of new plant is unchanged at

$$b = \frac{1}{R} V(1) = \frac{a - w^o}{R - \lambda}.$$

The budget constraint of the agent becomes

$$c_t + (x + q - b - s)i_t + \frac{d_{t+1}}{R} = wh_t + d_t.$$

Solving for the individuals' choices and aggregating across agents, we get

$$(x + q - b - s)H_{t+1} = \beta\pi^E \left[w + \lambda(x + q - b - s)\right] H_t + \beta\pi^S D_t,$$

$$\frac{D_{t+1}}{R} = \beta(1-\pi^E)\left[w + \lambda(x+q-b-s)\right] H_t + \beta(1-\pi^S) D_t.$$

As in (22), the steady state growth rate becomes

$$G = \beta R^E \left[\pi^E + \frac{\pi^S(1-\pi^E)R\beta}{G-(1-\pi^S)R\beta}\right], \tag{28}$$

where the rate of return for the engineer to invest with maximum leverage is

$$\begin{aligned} R^E &= \frac{w + \lambda(x+q-b-s)}{x+q-b-s} \\ &= \frac{(1-\tau)w^o}{x+q-b-\tau\frac{w^o}{G-\lambda}} + \lambda, \end{aligned}$$

using $(26, 27)$. Then we learn that the rate of return from investment changes with the tax and subsidy in the neighborhood of $\tau = 0$ as

$$\begin{aligned} \frac{dR^E}{d\tau} &= \frac{w}{x+q-b}\left[-1 + \frac{w}{(G-\lambda)(x+q-b)}\right] \\ &= \frac{w}{(x+q-b)(G-\lambda)}\left[\left(\frac{w}{x+q-b} + \lambda\right) - G\right] \\ &= \frac{w}{(x+q-b)(G-\lambda)}(R^E - G). \end{aligned} \tag{29}$$

Because the growth rate of the economy G is the weighted average of the growth rate of engineers, βR^E, and savers, βR, where $R^E > R$ in our economy, we learn $G < \beta R^E < R^E$ and

$$\frac{dR^E}{d\tau} > 0.$$

The equilibrium growth rate in (28) solves

$$\beta R^E = \frac{G}{\pi^E + \frac{\pi^S(1-\pi^E)\beta R}{G-(1-\pi^S)\beta R}}.$$

Since the RHS is an increasing function of G, we have

$$\frac{dG}{d\tau} > 0.$$

Thus the introduction of this tax and subsidy scheme increases steady state investment, and therefore growth, relative to the laissez-faire.

To get a handle on the overall effect of this policy intervention on the welfare of the domestic economy, we define a measure of social welfare as the population-weighted average of the expected discounted utilities of engineers and savers. The point is that we need to account for any short-term losses (as well as gains) at the time the policy is introduced, in addition to the longer-term benefits of higher growth. We show in Appendix B that, by this measure, social welfare goes *up*.

Why? In our framework, because of the non-exclusivity constraint (an individual engineer can work for any plant owner without getting traced by her creditors ex post), the engineers each face a borrowing constraint ex ante at the time of investment. By taxing the payroll of the plant owners, the government in effect acts as a collective creditor – the receipts from which, when fed back to the engineers, subsidize investment. It is as if, through the government intervention, the engineers *as a group* promise to pay back a portion of each others' debt obligations.[22] Crucial to the effectiveness of this policy is the government's ability to keep an eye on all the various units of plant (presumed to be fixed in buildings), to tax the owners' payments to engineers, in a context where the identities of the engineers themselves cannot be traced.

[22] In our model, the burden of the payroll tax is entirely borne by the engineers, because plant owners face unchanged wage costs and plant prices. In this sense, it is the most favorite case for the tax-subsidy scheme to boost the growth rate. In more general model, the tax burden would be split between the engineers and plant owners.

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9 Appendix

9.1 Appendix A

In the main text, we assume that output is proportional to plant productivity. More generally, suppose that gross output $\widehat{y}$ of each unit of plant depends upon the plant productivity $\widehat{z}$ and unskilled labor l as

$$\widehat{y} = \widehat{a}\widehat{z}^{\alpha_1} l^{\alpha_2}, \text{ where } \alpha_1 + \alpha_2 \leq 1.$$

Suppose there is a competitive labor market for unskilled workers at wage rate w_l. Then we can define the gross profit per plant as

$$\begin{aligned} y &= \underset{l}{Max}\left(\widehat{a}\widehat{z}^{\alpha_1} l^{\alpha_2} - w_l l\right) \\ &= az, \end{aligned} \tag{30}$$

where

$$\begin{aligned} z &= \widehat{z}^{\frac{\alpha_1}{1-\alpha_2}}, \\ a &= (1-\alpha_2)\left(\frac{\alpha_2}{w_l}\right)^{\frac{\alpha_2}{1-\alpha_2}} \widehat{a}^{\frac{1}{1-\alpha_2}}. \end{aligned}$$

If the supply of unskilled labor is perfectly elastic, we can treat a as exogenous – this is the case of our model. (Otherwise, we need to take into account the general equilibrium effect on a through w_l.)

If plant productivity depends upon initial plant productivity and human capital of engineer h as

$$\widehat{z}' = \widehat{z}^{\theta} h^{\widehat{\eta}}, \text{ where } \theta + \widehat{\eta} \leq 1.$$

we can rewrite this as

$$z' = z^{\theta} h^{\eta}, \text{ where } \eta = \frac{\alpha_1}{1-\alpha_2}\widehat{\eta}. \tag{31}$$

Thus we obtain the formulation in the text: $(30, 31)$.

9.2 Appendix B

9.2.1 Individual Choice

An individual agent takes wage, building price, plant price and interest rate $\varpi = \{w, q, b, R\}$ as given. An engineer chooses consumption, gross investment on tools and financial assets (c, h', d') as a function of net worth n to maximize $V^E(n; w, b, R)$,

$$V^E(n;\varpi) = \underset{c,h',d'\geq 0}{Max} \left\{\ln c + \beta\left[\pi^E V^E(n';\varpi) + (1-\pi^E)V^S(n';\varpi)\right]\right\}, \quad (32)$$

subject to the budget constraint

$$c + (x+q-b)\,h' + \frac{d'}{R} = n, \text{ and } n' = [w + \lambda\,(x+q-b)]\,h' + d'.$$

Define the leveraged rate of return on investment as

$$R^E = \frac{w + \lambda\,(x+q-b)}{x+q-b}.$$

The first order conditions of the engineer's optimization problem are

$$\begin{aligned} \frac{1}{c} &\geq R^E\frac{\beta}{c'}, \text{ where } = \text{ holds if } h' > 0, \\ \frac{1}{c} &\geq R\frac{\beta}{c'}, \text{ where } = \text{ holds if } d' > 0. \end{aligned}$$

Thus if $R^E > R$, we have $d' = 0$, $(12a, 12b)$ and

$$n' = R^E \beta n. \quad (33)$$

A saver chooses consumption and financial assets (c, d') as a function of net worth n to maximize

$$V^S(n;\varpi) = \underset{c,d'\geq 0}{Max} \left\{\ln c + \beta\left[\pi^S V^E(n';\varpi) + (1-\pi^S)V^S(n';\varpi)\right]\right\} \quad (34)$$

subject to the budget constraint

$$c + \frac{d'}{R} = n, \text{ and } n' = d'.$$

Using the first order condition

$$\frac{1}{c} = R\frac{\beta}{c'},$$

we get $(13a, 13b)$ and

$$n' = R\beta n. \tag{35}$$

From these, we conjecture that the value functions of the engineer and the saver are given by

$$V^E(n;\varpi) = \nu^E(\varpi) + \frac{1}{1-\beta}\ln n, \tag{36a}$$

$$V^S(n;\varpi) = \nu^S(\varpi) + \frac{1}{1-\beta}\ln n. \tag{36b}$$

From $(12b, 13b, 33, 35)$, the conjecture is verified if and only if

$$\nu^E(\varpi) = \beta\pi^E\nu^E(\varpi) + \beta(1-\pi^E)\nu^S(\varpi) + \frac{\beta}{1-\beta}\ln R^E(\varpi) + \ln(1-\beta),$$

$$\nu^S(\varpi) = \beta\pi^S\nu^E(\varpi) + \beta(1-\pi^S)\nu^S(\varpi) + \frac{\beta}{1-\beta}\ln R + \ln(1-\beta),$$

when there is no change of $\varpi = \{w, q, b, R\}$ in the future. Then we get

$$\nu^E(\varpi) = \beta\frac{(1-\beta+\beta\pi^S)\ln\left(R^E(\varpi)\right) + \beta(1-\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln(1-\beta)}{1-\beta}, \tag{37}$$

$$\nu^S(\varpi) = \beta\frac{\beta\pi^S\ln\left(R^E(\varpi)\right) + (1-\beta\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln(1-\beta)}{1-\beta}. \tag{38}$$

The plant owner/saver's choice is given by value function (6) in the main text. The first order condition for those who choose to continue operating the plant this period is

$$w \geq \eta\frac{z'}{h}\frac{\lambda}{R}V'(z';\varsigma), \text{ where } = \text{holds if } h > 0, \tag{39}$$

$$V'(z;\varsigma) = a + \theta\frac{z'}{z}\frac{\lambda}{R}V'(z';\varsigma). \tag{40}$$

where $\varsigma = \{w, q, R\}$ is aggregate state variables that the plant owner takes as given. From these, if $h_t, h_{t+1}, \ldots > 0$, we have

$$\begin{aligned} w &= \frac{\lambda}{R}\left[\eta \frac{z_{t+1}}{h_t} a + \eta \frac{z_{t+1}}{h_t} \theta \frac{z_{t+2}}{z_{t+1}} \frac{\lambda}{R} V'(z_{t+2}; \varsigma)\right] \\ &= \frac{\lambda}{R} a\eta \frac{z_{t+1}}{h_t} + \left(\frac{\lambda}{R}\right)^2 a\eta \frac{z_{t+1}}{h_t} \theta \frac{z_{t+2}}{z_{t+1}} + \left(\frac{\lambda}{R}\right)^3 a\eta \frac{z_{t+1}}{h_t} \theta \frac{z_{t+2}}{z_{t+1}} \theta \frac{z_{t+3}}{z_{t+2}} \\ &\quad + \ldots \end{aligned}$$

This is (14) in the text. Multiplying through by h_t, and simplifying, we get (15) in the text. Then we get

$$\begin{aligned} & V(z; \varsigma) \\ &= (y_t - wh_t) + \frac{\lambda}{R}(y_{t+1} - wh_{t+1}) + \frac{\lambda^2}{R^2}(y_{t+2} - wh_{t+2}) + \ldots \\ &= y_t + \frac{\lambda}{R} y_{t+1}(1 - \eta) + \frac{\lambda^2}{R^2} y_{t+2}(1 - \eta - \eta\theta) + \ldots \end{aligned}$$

This implies (16) in the text.

If $h_t, h_{t+1} > 0$, we can use (39, 40) to derive an alternative first order condition as

$$\begin{aligned} w &= \frac{\lambda}{R} \eta \frac{z_{t+1}}{h_t} a + \frac{\lambda}{R} w\eta \frac{z_{t+1}}{h_t} \frac{\theta \frac{z_{t+2}}{z_{t+1}}}{\eta \frac{z_{t+2}}{h_{t+1}}} \\ &= \frac{\lambda}{R} \eta \frac{z_{t+1}}{h_t} a + \frac{\lambda}{R} \theta \frac{h_{t+1}}{h_t} w. \end{aligned} \tag{41}$$

Note that the second term on the RHS equals the discounted wage rate times the marginal rate of substitution between h_t and h_{t+1} to keep z_{t+2} constant. Thus equation (41) says the marginal cost of increasing h_t by one unit equals the discounted value of marginal benefit – the sum of additional output through z_{t+1} and saving of wage bill, keeping z_{t+2} constant.

In the case of constant-returns-to-scale maintenance technology, $\theta + \eta = 1$, we conjecture

$$S^\infty(z; \varsigma) = aA^\infty z,$$

$$S^T(z; \varsigma) = aA^T z + \frac{\lambda^{T+1}}{R^T} q.$$

For plant which continues forever, we conjecture and verify that

$$\frac{h_{t+1}}{h_t} = \frac{z_{t+1}}{z_t} = \left(\frac{h_t}{z_t}\right)^{1-\theta} = g > 1.$$

Then from (41), we get

$$w = \frac{\frac{\lambda}{R}\eta\frac{z_{t+1}}{h_t}a}{1 - \frac{\lambda}{R}\theta g} = \frac{\lambda\left(1-\theta\right)a}{R - \lambda\theta g}g^{-\frac{\theta}{1-\theta}}. \tag{42}$$

Then from (6) , we learn that the Bellman equation for continuing plant holds if and only if

$$A^{\infty} = \frac{R}{R - \lambda\theta g}. \tag{43}$$

For stopping plant in finite time, (39) implies that

$$w = (1-\theta)\left(\frac{z^T}{h^T}\right)^{\theta}\frac{\lambda}{R}aA^{T-1}, \tag{44}$$

where z^T and h^T are the productivity and tools of plant when it stops in T periods. Then from (6) , we learn that the Bellman equation for continuing plant holds if and only if

$$\begin{aligned} A^T &= 1 + \frac{\lambda}{R}\theta A^{T-1}\left(\frac{1-\theta}{w}\frac{\lambda}{R}aA^{T-1}\right)^{\frac{1-\theta}{\theta}} \\ &= 1 + \frac{\lambda}{R}\theta g A^{T-1}, \end{aligned} \tag{45}$$

for $T \geq 1$, using (42), and $A^0 = 1$.

When the maintenance technology is decreasing returns to scale, $\theta + \eta < 1$, we conjecture that the productivity of plant that continues forever will converge to a steady state productivity

$$z = z^*.$$

Thus the amount of tools employed converges to

$$h = h^* = (z^*)^{\frac{1-\theta}{\eta}}.$$

We also conjecture that

$$S^{\infty}(z;\varsigma) = az^{*}U^{\infty}\left(\frac{z}{z^{*}};R\right),$$

$$S^{T}(z;\varsigma) = az^{*}U^{T}\left(\frac{z}{z^{*}};R\right) + \frac{\lambda^{T+1}}{R^{T}}q.$$

Using (41) for plant to continue forever in steady state, we get

$$w = \frac{\lambda\eta a}{R-\lambda\theta}(z^{*})^{-\frac{1-\eta-\theta}{\eta}}. \tag{46}$$

Define $\widetilde{z} = \frac{z}{z^{*}}$. Using the relationship $h = \left(\frac{z'}{z^{\theta}}\right)^{\frac{1}{\eta}}$, we get

$$\frac{wh}{az^{*}} = \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}}.$$

Thus the guess is verified if $U^{\infty}(\widetilde{z})$ and $U^{T}(\widetilde{z})$ solve

$$U^{\infty}(\widetilde{z};R) = \underset{\widetilde{z}'}{Max}\,\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}} + \frac{\lambda}{R}U^{\infty}(\widetilde{z}';R), \tag{47}$$

$$U^{T}(\widetilde{z};R) = \underset{\widetilde{z}'}{Max}\,\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}} + \frac{\lambda}{R}U^{T-1}(\widetilde{z}';R), \tag{48}$$

for $T \geq 1$ and $U^{0}(\widetilde{z};R) = \widetilde{z}$.

9.2.2 Market Clearing

In order to describe the aggregate economy, let $K_t(\tau)$ be the aggregate number of age-τ plant which continues forever at date t. Suppose some owners choose to operate new plant for T periods and then stop. Let $L_t^{T-\tau}(\tau)$ be aggregate number of age-τ plant which stops in $T-\tau$ periods at date t. A plant that stops in 0 period stops at the end of the period, after production. Then the laws of motion for $K_t(\tau)$ and $L_t^{T-\tau}(\tau)$ are

$$\begin{aligned} K_t(\tau) &= \lambda K_{t-1}(\tau-1), \\ L_t^{T-\tau}(\tau) &= \lambda L_{t-1}^{T-\tau+1}(\tau-1), \text{ for } \tau = 1,2,\ldots,T. \end{aligned} \tag{49}$$

Let I_t be the aggregate investment at date t. We also have

$$I_t = K_{t+1}(0) + L_{t+1}^{T}(0), \tag{50}$$

We also know that

$$\begin{aligned} b &= \frac{1}{R}S^{\infty}(1;\varsigma) = \frac{1}{R}S^{T}(1;\varsigma) \text{ in M-Region,} \\ b &= \frac{1}{R}S^{\infty}(1;\varsigma) \text{ and } L_t^{T}(0) = 0 \text{ in P-Region.} \end{aligned} \tag{51}$$

Let $z_t^{T-\tau}(\tau)$ be the productivity of age-τ plant which stops in $T-\tau$ periods at date t. Let $h_t^{T-\tau}(\tau)$ be the number of tools employed by one unit of age-τ plant to stop in $T-\tau$ periods. Then the aggregate output and demand for tools (and engineers) are given by

$$Y_t = \sum_{\tau=0}^{\infty} a z_t^{\infty}(\tau)K_t(\tau) + \sum_{\tau=0}^{T} a z_t^{T-\tau}(\tau)L_t^{T-\tau}(\tau), \tag{52}$$

$$H_t = \sum_{\tau=0}^{\infty} h_t^{\infty}(\tau)K_t(\tau) + \sum_{\tau=0}^{T} h_t^{T-\tau}(\tau)L_t^{T-\tau}(\tau). \tag{53}$$

Aggregate domestic asset holding at the beginning of period t equals the sum of gross profit and the value of plant from the last period minus net foreign debt:

$$\begin{aligned} D_t &= Y_t - wH_t - D_t^{*} \\ &\quad + \frac{1}{R}\left[\sum_{\tau=1}^{\infty} V\left(z\left(\tau\right)\right)K_t(\tau) + \sum_{\tau=1}^{T} S^{T-\tau}\left(z_t^{T-\tau}(\tau)\right)L_t^{T-\tau}(\tau)\right]. \end{aligned} \tag{54}$$

The goods market clearing condition is given by

$$C_t + (x+q)I_t + D_t^{*} - \frac{D_{t+1}^{*}}{R} = Y_t. \tag{55}$$

Output equals consumption, investment and net export (which equals net debt repayment to foreigners). One of the market clearing conditions for output, tools and financial asset is not independent by Walras' Law.

9.2.3 Pure Equilibrium with No Stopping

When no plant owner stops his plant, the total number of continuing plant equals the total number of tools,

$$\sum_{\tau=0}^{\infty} K_t(\tau) = H_t,$$

and the ratio of tools to plant remains at the initial ratio

$$h_t^{\infty}(\tau) = 1, \text{ for all } \tau \text{ and } t.$$

The plant productivity remains at the initial level as

$$z_t^{\infty}(\tau) = \left[z_{t-1}^{\infty}(\tau-1)\right]^{\theta}\left[h_{t-1}^{\infty}(\tau-1)\right]^{\eta} = 1, \text{ for all } \tau \text{ and } t.$$

Thus the plant growth rate $g = 1$ with constant-returns-to-scale maintenance technology, steady state productivity $z^* = 1$ with decreasing-returns-to-scale maintenance technology, and

$$w = \frac{\lambda \eta a}{R - \lambda\theta} = w(R), \tag{56a}$$

$$b = \frac{a-w}{R-\lambda} = \frac{a}{R-\lambda}\frac{R-\lambda(\theta+\eta)}{R-\lambda\theta} = b(R). \tag{56b}$$

In order to show that non-stopping is an optimal strategy for the plant owner, we need to check

$$b(R) > \frac{1}{R}\underset{T}{Max} S^T(1;\varsigma(R)) = \frac{1}{R}\underset{T}{Max}[aU^T(1;\varsigma(R)) + \frac{\lambda^{T+1}}{R^T}q(R)], \tag{57}$$

for any finite T, where $U^T(1;R)$ is given by (48) with decreasing returns to scale and equals A^T with the constant returns to scale maintenance technology, and $\varsigma(R) = \{w(R), q(R), R\}$ takes into consideration the dependence of wage rate and building price on interest rate R.

Then from (52, 54), we have

$$\begin{aligned} Y_t &= aH_t, \\ D_t &= (a-w)H_t + b\lambda H_t - D_t^*. \end{aligned} \tag{58}$$

We also have

$$C_t = (1-\beta)[(w + \lambda(x+q-b))H_t + D_t]. \tag{59}$$

From $(21a, 55)$, we obtain the transitions:

$$
\begin{aligned}
(x+q-b)H_{t+1} &= \beta\left\{\pi^E(w+\lambda(x+q-b))H_t + \pi^S D_t\right\}, && (60a)\\
\frac{D^*_{t+1}}{R} &= -aH_t + C_t + (x+q)(H_{t+1}-\lambda H_t) + D^*_t. && (60b)
\end{aligned}
$$

(w, q, b) is a function of R and the other parameters, and (D_t, C_t) is a function of (H_t, D^*_t) and R (through w and b). Then, the perfect foresight equilibrium (aside from a unanticipated permanent shock on R) is characterized recursively by $\left(H_{t+1}, D^*_{t+1}\right)$ as a function of (H_t, D^*_t, R).

In steady state, we can use (22) to find steady-state growth rate where

$$
R^E = \frac{w(R) + \lambda\left[x + q(R) - b(R)\right]}{x + q(R) - b(R)}.
$$

9.2.4 Mixed Equilibrium

For the mixed equilibrium, we only describe the steady state equilibrium.

Mixed equilibrium under constant returns to scale maintenance technology From $(42, 43)$, we have

$$
\begin{aligned}
w &= \frac{\lambda(1-\theta)a}{R-\lambda\theta g} g^{-\frac{\theta}{1-\theta}} = w(g; R)\\
b &= \frac{a}{R-\lambda\theta g} = b(g; R).
\end{aligned}
$$

Find $\left\{A^1, A^2, A^3, \ldots, A^T\right\}$ to solve (45) with $A^0 = 1$ as a function of $(g; R)$. Find g to solve the indifference condition:

$$
b(g; R) = \frac{1}{R} \underset{\text{finite } T}{Max} \left[aA^T(g; R) + \frac{\lambda^{T+1}}{R^T} q(R)\right]. \tag{61}
$$

Equilibrium stopping time is the solution to equation (61) for this equilibrium g.

Then we can find the steady state growth rate from (22) by using

$$
R^E = \frac{w(g; R) + \lambda[x + q(R) - b(g; R)]}{x + q(R) - b(g; R)}.
$$

For plant that continues forever, because $z^{\infty}(0) = 1$, we get $z^{\infty}(\tau) = g^{\tau}$ and

$$h^{\infty}(\tau) = \left[\frac{z^{\infty}(\tau+1)}{(z^{\infty}(\tau))^{\theta}}\right]^{\frac{1}{1-\theta}} = g^{\frac{1}{1-\theta}+\tau}.$$

For those stopping in T periods, we get from the first order condition (44)

$$\frac{h^{T-\tau}(\tau)}{z^{T-\tau}(\tau)} = \left[\frac{(1-\theta)\lambda}{w/a}A^{T-\tau-1}\right]^{\frac{1}{\theta}} = \left(\frac{A^{T-\tau-1}}{A^{\infty}}\right)^{\frac{1}{\theta}} g^{\frac{1}{1-\theta}}, \tag{62}$$

for $\tau = 0, 1, 2, \ldots, T-1$. Because $z^{T}(0) = 1$, we obtain $\{h^{T-\tau}(\tau), z^{T-\tau-1}(\tau+1)\}$ which satisfies (62) and

$$z^{T-\tau-1}(\tau+1) = \left(\frac{A^{T-\tau-1}}{A^{\infty}}\right)^{\frac{1-\theta}{\theta}} g z^{T-\tau}(\tau),$$

for $\tau = 0, 1, 2, \ldots, T-1$.

Mixed equilibrium under decreasing returns to scale maintenance technology With decreasing returns, from (46), we get

$$w = \frac{\lambda\eta a}{R-\lambda\theta}(z^{*})^{-\frac{1-\theta-\eta}{\eta}} = w(z^{*}; R).$$

For plant to continue for ever, we have from (47):

$$\begin{aligned} U^{\infty}(\widetilde{z}) &= \underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}} + \frac{\lambda}{R}U^{\infty}(\widetilde{z}')\right] \\ \widetilde{z}' &= \arg\underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}} + \frac{\lambda}{R}U^{\infty}(\widetilde{z}')\right] \equiv \varphi^{\infty}(\widetilde{z}) \end{aligned}$$

Let $\widetilde{z}^{\infty}(\tau)$ and $\widetilde{h}^{\infty}(\tau)$ be productivity and number of tools of age-τ plant which continues forever relative to the steady state. The we have

$$\begin{aligned} \widetilde{z}^{\infty}(\tau) &= (\varphi^{\infty})^{\tau}(\widetilde{z}^{\infty}(0)) = (\varphi^{\infty})^{\tau}\left(\frac{1}{z^{*}}\right) \\ \widetilde{h}^{\infty}(\tau) &= \left[\frac{\widetilde{z}^{\infty}(\tau+1)}{(\widetilde{z}^{\infty}(\tau))^{\theta}}\right]^{\frac{1}{\eta}}. \end{aligned}$$

For plant to stop in T periods, we have from (48):

$$\begin{aligned} U^T(\widetilde{z}) &= \underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}} + \frac{\lambda}{R}U^{T-1}(\widetilde{z}')\right] \\ \widetilde{z}' &= \arg\underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^{\theta}}\right)^{\frac{1}{\eta}} + \frac{\lambda}{R}U^{T-1}(\widetilde{z}')\right] \equiv \varphi^T(\widetilde{z}) \ , \end{aligned}$$

where $U^0(\widetilde{z}) = \widetilde{z}$. Let $\widetilde{z}^{T-\tau}(\tau)$ and $\widetilde{h}^{T-\tau}(\tau)$ be productivity and tools of age-τ plant which stops in $T-\tau$ periods relative to the steady state. Then we have

$$\begin{aligned} \widetilde{z}^{T-\tau}(\tau) &= \varphi^T \cdot \varphi^{T-1} \cdot \ldots \cdot \varphi^{T-\tau+1}\left(\frac{1}{z^*}\right) \\ \widetilde{h}^{T-\tau}(\tau) &= \left[\frac{\widetilde{z}^{T-\tau-1}(\tau+1)}{(\widetilde{z}^{T-\tau}(\tau))^{\theta}}\right]^{\frac{1}{\eta}}. \end{aligned}$$

We then find z^* to satisfy the indifference condition

$$\begin{aligned} az^*U^{\infty}\left(\frac{1}{z^*}\right) &= \underset{\text{finite } T}{Max}\left[az^*U^T\left(\frac{1}{z^*}\right) + \frac{\lambda^{T+1}}{R^T}q(R)\right] && \text{(63a)} \\ &= Rb\,(z^*; R) && \text{(63b)} \end{aligned}$$

This common value under equilibrium z^* is the engineer's borrowing capacity. Equilibrium stopping time equals $\arg Max\left[az^*U^T\left(\frac{1}{z^*}\right) + \frac{\lambda^{T+1}}{R^T}q(R)\right]$.

We can find the steady state growth rate from (22) with

$$R^E = \frac{w(z^*; R) + \lambda\left[x + q(R) - b\,(z^*; R)\right]}{x + q(R) - b\,(z^*; R)} = R^E\,(z^*; R)\,.$$

9.2.5 Tools and goods market clearing in mixed equilibrium

In the steady state, we observe

$$G = \frac{H_{t+1}}{H_t} = \frac{K_{t+1}(\tau)}{K_t\,(\tau)} = \frac{L_{t+1}^{T-\tau}(\tau)}{L_t^{T-\tau}(\tau)}.$$

For both constant and decreasing returns-to-scale maintenance technology, we have aggregate output under mixed equilibrium as (52). Using (49), we obtain

$$Y_t = \sum_{\tau=0}^{\infty} az^{\infty}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}K_t\,(0) + \sum_{\tau=0}^{T} az^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}L_t^T\,(0)\,.$$

Similarly, aggregate demand for tools (53) becomes

$$H_t = \sum_{\tau=0}^{\infty} h^{\infty}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}K_t(0) + \sum_{\tau=0}^{T} h^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}L_t^T(0). \tag{64}$$

Because $I_t = (G-\lambda)H_t = K_{t+1}(0) + L_{t+1}^T(0)$, dividing (64) by H_t, we find in the steady state:

$$1 = \sum_{\tau=0}^{\infty} h^{\infty}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k + \sum_{\tau=0}^{T} h^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k), \tag{65}$$

where $i^k \equiv \frac{K_{t+1}(0)}{I_t} \in (0,1)$. We can solve for $i^k \in (0,1)$ to satisfy (65). Similarly, output per tool is

$$\frac{Y_t}{H_t} = \sum_{\tau=0}^{\infty} az^{\infty}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k + \sum_{\tau=0}^{T} az^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k). \tag{66}$$

Aggregate domestic financial asset holding (54) under constant-returns-to-scale maintenance technology is given by

$$D_t = Y_t - wH_t - D_t^*$$

$$+\sum_{\tau=1}^{\infty}\frac{a}{R-\lambda\theta g}\frac{\lambda^{\tau}}{G^{\tau}}K_t(0) + \sum_{\tau=1}^{T} aA^{T-\tau}z^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}L_t^T(0),$$

or

$$\frac{D_t}{H_t} = \frac{Y_t}{H_t} - w - d_t^*$$

$$+\sum_{\tau=1}^{\infty}\frac{a}{R-\lambda\theta g}\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k + \sum_{\tau=1}^{T} aA^{T-\tau}z^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k),$$

where $d_t^* = D_t^*/H_t$.

Similarly, domestic financial asset holding per tool under decreasing returns to scale is

$$\frac{D_t}{H_t} = \frac{Y_t}{H_t} - w - d_t^*$$

$$+\frac{1}{R}\left[\sum_{\tau=1}^{\infty} az^*U\left(\widetilde{z}^{\infty}(\tau)\right)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k + \sum_{\tau=1}^{T} az^*U\left(\widetilde{z}^{T-\tau}(\tau)\right)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k)\right].$$

We also find

$$\frac{C_t}{H_t} = (1-\beta)\left[w + \lambda(x+q-b) + \frac{D_t}{H_t}\right].$$

From (55) ,in steady state,

$$\frac{Y_t}{H_t} = \frac{C_t}{H_t} + (x+q)(G-\lambda) + d^* - \frac{G}{R}d^*$$

or

$$\left(1-\frac{G}{R}\right)d^* = \frac{Y_t}{H_t} - \frac{C_t}{H_t} - (x+q)(G-\lambda). \tag{67}$$

From this, we find the ratio of net foreign debt to tools in steady state.

9.3 Proof of Proposition 1

We derive a sufficient condition for the existence of a pure non-stopping equilibrium in P-region:

$$\begin{aligned} V(1) &= \frac{R}{R-\lambda}a\frac{R-(\theta+\eta)\lambda}{R-\theta\lambda} \\ &\geq a\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\frac{1-\theta-\eta}{1-\theta} + \frac{a\eta R}{(1-\theta)(R-\theta\lambda)} + \frac{\lambda f}{R-\lambda}. \end{aligned} \tag{68}$$

We consider a sufficient condition of (57)

$$b(R) > \frac{1}{R}\underset{T}{Max}S^T(1; w(R), R),$$

for the case of decreasing-returns-to-scale maintenance technology. Consider an optimal stopping strategy where the plant owner stops in T periods in the RHS as

$$\left\{z^T(0) > z^{T-1}(1) > \ldots > z^0(T)\right\} = \{z_0 > z_1 > \ldots > z_T\}$$

such that $z_0 = 1$ and $z_T \geq \underline{z} = f/a$. Associated with $\{z_t\}$, there is a sequence of human capital demand $h_t = \left(\frac{z_{t+1}}{z_t^\theta}\right)^{1/\eta}$. Let $v(h|z)$ denote the flow payoff of the owner of a unit of plant with productivity z who hires h units of tools.

$$v(h|z) = az - wh.$$

Because optimal stopping strategy $z_t > z_{t+1}$ is better than staying at z_t with $h = z_t^{\frac{1-\theta}{\eta}}$, we get

$$v(h_t|z_t) + \frac{\lambda}{R}V(z_{t+1}) \geq v\left(z_t^{\frac{1-\theta}{\eta}}|z_t\right) + \frac{\lambda}{R}V(z_t), \text{ or}$$

$$V(z_t) - V(z_{t+1}) \leq \frac{R}{\lambda}\left[v(h_t|z_t) - v\left(z_t^{\frac{1-\theta}{\eta}}|z_t\right)\right]. \tag{69}$$

Let $\phi(z|z_t) \equiv v\left(\left(z/z_t^{\theta}\right)^{\frac{1}{\eta}}\middle| z_t\right) = az_t - w\left(z/z_t^{\theta}\right)^{\frac{1}{\eta}}$.

$$v(h_t|z_t) - v(z_t^{\frac{1-\theta}{\eta}}|z_t) = \int_{z_{t+1}}^{z_t} -\phi'(z|z_t)dz,$$

where

$$-\phi'(z|z_t) = \frac{w}{\eta}\frac{z^{\frac{1}{\eta}-1}}{z_t^{\frac{\theta}{\eta}}}.$$

Notice that because

$$\frac{\partial}{\partial z_t}\left[-\phi'(z|z_t)\right] < 0,$$

we have

$$-\phi'(z|z_t) = \frac{w}{\eta}\frac{z^{\frac{1}{\eta}-1}}{z_t^{\frac{\theta}{\eta}}} \leq \frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1} = -\phi'(z|z), \text{ for } z_{t+1} \leq z \leq z_t.$$

Then,

$$v(h_t|z_t) - v(z_t^{\frac{1-\theta}{\eta}}|z_t) = \int_{z_{t+1}}^{z_t} -\phi'(z|z_t)dz \leq \int_{z_{t+1}}^{z_t} -\phi'(z|z)dz.$$

Combining this inequality with inequality (69), we have

$$V(z_t) - V(z_{t+1}) \leq \frac{R}{\lambda}\left[v(h_t|z_t) - v\left(z_t^{\frac{1-\theta}{\eta}}|z_t\right)\right] \leq \frac{R}{\lambda}\int_{z_{t+1}}^{z_t}\frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1}dz,$$

$$V(1) - V(z_T) = \sum_{t=0}^{T-1}\left[V(z_t) - V(z_{t+1})\right] \leq \frac{R}{\lambda}\int_{z_T}^{1}\frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1}dz,$$

where we use $z_1 = 1$ in the last inequality. Because

$$V(z_T) = az_T + \lambda q$$

and

$$\frac{R}{\lambda}\int_{z_T}^{1}\frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1}dz = \frac{Rw}{\lambda(1-\theta)}\left(1 - z_T^{\frac{1-\theta}{\eta}}\right),$$

we have

$$V(1) \leq az_T + \lambda q + \frac{Rw}{\lambda(1-\theta)}\left(1 - z_T^{\frac{1-\theta}{\eta}}\right) \equiv RHS\left(z_T\right), \tag{70}$$

if we are not in region P, i.e., some plant owners stop their plant.

To derive a sufficient condition for Region P, we use the fact that equilibrium wage in this region satisfies

$$\frac{w}{a} = \frac{\lambda\eta}{R-\theta\lambda}.$$

Then RHS of (70) reaches the maximum when

$$\begin{aligned} z_T &= \left(1 - \frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}} \\ RHS &= a\left(1 - \frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\frac{1-\theta-\eta}{1-\theta} + \frac{a\eta R}{(1-\theta)\left(R-\theta\lambda\right)} + \lambda q. \end{aligned}$$

A sufficient condition for the economy to be in Region P is

$$\begin{aligned} V(1) &= \frac{aR}{R-\lambda}\frac{R-(\theta+\eta)\lambda}{R-\theta\lambda} \\ &\geq a\left(1 - \frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\frac{1-\theta-\eta}{1-\theta} + \frac{a\eta R}{(1-\theta)\left(R-\theta\lambda\right)} + \lambda\frac{f}{R-\lambda}. \end{aligned}$$

This yields an upper bound on f/a:

$$\frac{f}{a} \leq \frac{R\left(1-\theta-\eta\right)}{\lambda(1-\theta)}\left[1 - \frac{R-\lambda}{R}\left(1 - \frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\right] \equiv \overline{F}(f/a).$$

$\overline{F}(f/a)$ denotes an upper bound for f/a as a sufficient condition for the existence of a pure equilibrium with no stopping.

9.4 Proof of Proposition 3

We first derive a lower bound on f/a such that the growth rate is an increasing function of real interest rate in state equilibrium. From (22) , we learn

$$\begin{aligned} 0 &= (G-\pi^E\beta R^E)[G-(1-\pi^S)\beta R]-\pi^S(1-\pi^E)\beta^2 RR^E \\ &= \left[G-\pi^E\beta\left(\lambda+\frac{w}{x+q-b}\right)\right][G-(1-\pi^S)\beta R]-\pi^S(1-\pi^E)\beta^2 R\left(\lambda+\frac{w}{x+q-b}\right) \\ &\equiv \Psi\left(G;R,\frac{w}{x+q-b}\right). \end{aligned} \tag{71}$$

Because we assume $\beta R<1$, we restrict our attention the case

$$G>(1-\pi^S)\beta R.$$

Then we learn

$$G\geq \pi^E\beta\left(\lambda+\frac{w}{x+q-b}\right).$$

Then we learn

$$\frac{\partial}{\partial G}\Psi\left(G;R,\frac{w}{x+q-b}\right)>0,$$

in the neighborhood of the equilibrium G. We can easily check

$$\frac{\partial}{\partial R}\Psi\left(G;R,\frac{w}{x-b}\right)<0,$$

$$\frac{\partial}{\partial\left(\frac{w}{x+q-b}\right)}\Psi\left(G;R,\frac{w}{x+q-b}\right)<0.$$

Thus a sufficient condition for

$$\frac{dG}{dR}=-\frac{\frac{\partial}{\partial G}\Psi\left(G;R,\frac{w}{x+q-b}\right)}{\frac{\partial}{\partial R}\Psi\left(G;R,\frac{w}{x+q-b}\right)+\frac{\partial}{\partial\left(\frac{w}{x+q-b}\right)}\Psi\left(G;R,\frac{w}{x+q-b}\right)\frac{d}{dR}\left(\frac{w}{x+q-b}\right)}>0$$

is

$$\begin{aligned} 0 &< \frac{d}{dR}\left(\frac{w}{x+q-b}\right) \\ &= \frac{w}{(x+q-b)^2(R-\lambda)^2(R-\lambda\theta)}\left[\lambda(1-\theta)f-(R-\lambda)^2x-\lambda(1-\theta-\eta)a\right], \end{aligned}$$

or

$$\lambda\,(1-\theta)\,f > (R-\lambda)^2 x + \lambda\,(1-\theta-\eta)\,a. \tag{72}$$

If $\pi^S = 0$, then from (22), we have $G = \pi^E\beta\left(\lambda + \frac{w}{x+q-b}\right)$, or

$$\begin{aligned} x = F(R,G) &= \frac{a-f-w}{R-\lambda} + \frac{\beta\pi^E}{G-\beta\lambda\pi^E}w \\ &= \frac{a-f}{R-\lambda} - \frac{G-\beta R\pi^E}{(R-\lambda)\,(G-\beta\lambda\pi^E)}w. \end{aligned}$$

Because $F_G < 0$, $dG/dR > 0$ if and only if $F_R > 0$. And because

$$(R-\lambda)F_R = -\frac{a-f-w}{R-\lambda} + \frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{a\eta\lambda}{(R-\theta\lambda)^2},$$

$dG/dR > 0$ iff

$$f/a > \frac{R-(\theta+\eta)\lambda}{R-\theta\lambda} - \frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{\eta\lambda\,(R-\lambda)}{(R-\theta\lambda)^2} \equiv \underline{F}(f/a)$$

when $\pi^S = 0$.

Next, for the growth-enhancing effect of interest rate to be in Region P, we need

$$\overline{F}(f/a) - \underline{F}(f/a) > 0$$

or

$$\begin{aligned} \frac{\overline{F}(f/a) - \underline{F}(f/a)}{R-\lambda} &= \frac{R\,(1-\theta-\eta) - \lambda(1-\theta)(\theta+\eta)}{\lambda(1-\theta)\,(R-\theta\lambda)} \\ &\quad - \frac{1-\theta-\eta}{\lambda(1-\theta)}\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}} + \frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{\eta\lambda}{(R-\theta\lambda)^2} > 0. \end{aligned}$$

Suppose both R and λ are close to 1,

$$\begin{aligned} \frac{\overline{F}(f/a) - \underline{F}(f/a)}{R-\lambda} &\approx \frac{1-\theta-\eta-(1-\theta)(\theta+\eta)}{(1-\theta)^2} - \frac{1-\theta-\eta}{(1-\theta)}(1-\theta)^{\frac{\eta}{1-\theta-\eta}} + \frac{\eta}{(1-\theta)^2} \\ &= \frac{1-\theta-\eta}{1-\theta}\left[1-(1-\theta)^{\frac{\eta}{1-\theta-\eta}}\right] \\ &> 0. \end{aligned}$$

This proves that for any f/a, there exists an open set of interest rates and depreciation rates, both of which are close to 1, where we have the property that the growth rate is an increasing function of the interest rate in Region P.

To examine the effect of an unanticipated fall in real interest rate on welfare in the pure non-stopping region, we use $(36a, 36b, 37, 38)$. Continue to assume $\pi^S = 0$. Then we have

$$\begin{aligned}
\frac{dV^E}{dR} &= \frac{1}{1-\beta}\frac{d}{dR}(\ln n^E) \\
&\quad + \frac{\beta}{(1-\beta)(1-\beta\pi^E)}\frac{d}{dR}\left[\ln\left(\frac{w+\lambda(x+q-b)}{x+q-b}\right)\right] \\
&\quad + \frac{\beta^2(1-\pi^E)}{(1-\beta)^2(1-\beta\pi^E)}\frac{d}{dR}\ln R. \qquad (73)
\end{aligned}$$

From $(56a, 56b)$, we have

$$\begin{aligned}
\frac{dw}{dR} &= -\frac{w}{R-\lambda\theta}, \\
\frac{db}{dR} &= \frac{1}{R-\lambda}\left(\frac{w}{R-\lambda\theta} - b\right).
\end{aligned}$$

Then we get

$$\begin{aligned}
\frac{d}{dR}\ln[w+\lambda(x+q-b)] &= \frac{1}{w+\lambda(x+q-b)}\frac{1}{(R-\lambda)^2}\left(a-f-\frac{R^2-\lambda^2\theta}{(R-\lambda\theta)^2}\eta a\right), \\
\frac{d}{dR}\ln\left(\lambda+\frac{w}{x+q-b}\right) &= \frac{w}{[w+\lambda(x+q-b)](x+q-b)(R-\lambda)^2(R-\lambda\theta)^2} \\
&\quad \cdot\left[\lambda\eta a - \lambda(1-\theta)(a-f) - (R-\lambda)^2 x\right].
\end{aligned}$$

When $\pi^S = 0$, $n^E = [w+\lambda(x+q-b)]\,h$. Then from (73), we have

$$\begin{aligned}
& (1-\beta)(1-\beta\pi^E)(R-\lambda)^2(R-\lambda\theta)^2[w+\lambda(x+q-b)]\frac{dV^E}{dR}/\lambda \\
= \; & (1-\beta\pi^E)\left[(R-\lambda\theta)^2(a-f)-(R^2-\lambda^2\theta)\eta a\right] \\
& + \frac{\beta a\eta}{x+q-b}\left[\lambda\eta a - \lambda(1-\theta)(a-f)-(R-\lambda)^2 x\right] \\
& + \frac{\beta^2(1-\pi^E)}{1-\beta}\frac{(R-\lambda)(R-\lambda\theta)}{R}\{(R-\lambda\theta)[(R-\lambda)x-(a-f)]+R\eta a\}.
\end{aligned}$$

Then we can find an open set of the P-Region in which $\frac{dV^E}{dR} > 0$ following an anticipated change of R.

When $\pi^S = 0$, we know $(36b, 36b)$ that

$$\begin{aligned}(1-\beta)\,V^S &= \frac{\beta}{1-\beta}\ln R + \ln n^S + \text{constant, where} \\ n^S &\propto (a-w+\lambda b)H - D^*.\end{aligned}$$

Define

$$l^* = \frac{D^*}{(a-w+\lambda b)H}$$

as the leverage rate of this country. Then

$$(1-\beta)\,\frac{dV^S}{dR} = \frac{\beta}{1-\beta}\frac{1}{R} + \frac{1}{1-l^*}\frac{Rw-\lambda(R-\lambda\theta)b}{(R-\lambda)(R-\lambda\theta)(a-w+\lambda b)},$$

or

$$\begin{aligned}&\frac{dV^S}{dR}\,(1-\beta)^2\,(1-l^*)\,(R-\lambda)(R-\lambda\theta)(a-w+\lambda b)/a \\ = \;& \beta\,(1-l^*)\,[R-\lambda\,(\theta+\eta)] + \lambda(1-\beta)\frac{\left(R^2-\lambda^2\theta\right)\eta-(R-\lambda\theta)^2}{(R-\lambda)(R-\lambda\theta)}\end{aligned}$$

Then we can find an open set of parameters in P-Region in which $\frac{dV^S}{dR} > 0$. Therefore, we can find an open subset of the P-Region, all agents is strictly worse off immediately after an unexpected fall in the real interest rate.

9.4.1 Welfare effect of policy

From $(36a, 36b, 37, 38)$, we learn that the welfare of a continuing engineer, retiring engineer, new engineer, and continuing saver are

$$\begin{aligned}V^{EE} &= \beta\frac{(1-\beta+\beta\pi^S)\ln R^E + \beta(1-\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln\{[w+\lambda\,(x+q-b-s)]h\}}{1-\beta} + \text{constant} \\ V^{ES} &= \beta\frac{\beta\pi^S\ln R^E + (1-\beta\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln\{[w+\lambda\,(x+q-b-s)]h\}}{1-\beta} + \text{constant} \\ V^{SE} &= \beta\frac{(1-\beta+\beta\pi^S)\ln R^E + \beta(1-\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln d}{1-\beta} + \text{constant} \\ V^{SS} &= \beta\frac{\beta\pi^S\ln R^E + (1-\beta\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln d}{1-\beta} + \text{constant,}\end{aligned}$$

where h is the number of tools and d is financial asset held from the last period. Notice that government tax-subsidy does not affect the value of plant b and thus it does not affect d. From $(26, 27, 29)$, we see that in the neighborhood of $\tau = 0$,

$$\frac{d}{d\tau} \ln R^E = \frac{w}{w + \lambda(x + q - b)} \cdot \frac{R^E - G}{G - \lambda},$$

$$\frac{d}{d\tau} \ln \left[w + \lambda\left(x + q - b - s\right)\right] = -\frac{w}{w + \lambda(x + q - b)} \cdot \frac{G}{G - \lambda}.$$

In steady state, we learn that the fractions of population of engineers and savers, (m_E, m_S), satisfy

$$\pi^S m_S = (1 - \pi^E) m_E,$$

where the LHS is the flow of savers to become engineers and the RHS is the flow of retiring engineers. Thus

$$m_E = \frac{\pi^S}{\pi^S + 1 - \pi^E}, \text{ and } m_S = \frac{1 - \pi^E}{\pi^S + 1 - \pi^E}.$$

We consider a welfare measure as the population-weighted average of the welfare of each type of agents:

$$V = m_E \left[\pi^E V^{EE} + (1 - \pi^E) V^{ES}\right] + m_S \left[\pi^S V^{SE} + (1 - \pi^S) V^{SS}\right].$$

Using the above expressions, we learn

$$\begin{aligned} V &= \frac{\pi^S \nu^E + (1 - \pi^E)\nu^S}{\pi^S + 1 - \pi^E} + \frac{\pi^S \ln[w + \lambda(x + q - b)]}{\pi^S + 1 - \pi^E} + \text{constant} \\ &= \frac{\pi^S}{(\pi^S + 1 - \pi^E)(1 - \beta)^2} \left\{\beta \ln R^E + (1 - \beta) \ln[w + \lambda(x + q - b)]\right\} + \text{constant.} \end{aligned}$$

Therefore the effect of a tax and subsidy on the social welfare is

$$\begin{aligned} \frac{dV}{d\tau} &= \frac{\pi^S}{(\pi^S + 1 - \pi^E)(1 - \beta)^2} \frac{w}{w + \lambda(x + q - b)} \left[\beta \frac{R^E - G}{G - \lambda} - (1 - \beta) \frac{G}{G - \lambda}\right] \\ &= \frac{\pi^S}{(\pi^S + 1 - \pi^E)(1 - \beta)^2} \frac{w}{[w + \lambda(x + q - b)](G - \lambda)} (\beta R^E - G) \\ &> 0. \end{aligned}$$

The last inequality is obtained because the growth rate of economy is the weighted average of growth rate of engineers βR^E and savers βR and $R^E > R$ in our economy.

Individually, if π^S is close to zero, we learn

$$\frac{dV^{EE}}{d\tau} > 0, \frac{dV^{ES}}{d\tau} < 0, \frac{dV^{SE}}{d\tau} > 0, \frac{dV^{SS}}{d\tau} > 0.$$

For the continuing engineer, because the welfare gain from the higher rates of return dominates the loss from the lower new worth, welfare increases, $\frac{dV^{EE}}{d\tau} > 0$. For the retiring engineer, the loss from lower net worth dominates the gain from the higher rates of return when she becomes an engineer in the future, and thus welfare decreases, $\frac{dV^{ES}}{d\tau} < 0$. For those who were the savers in the previous period, there is no capital loss and only gains from the higher rates of return, and welfare increases, $\frac{dV^{SE}}{d\tau}, \frac{dV^{SS}}{d\tau} > 0$.

9.5 Calibration strategy

We choose the following parameter values, θ, η, λ, β, π^E and π^S. We normalize the productivity of plant productivity a to be 1.

We solve for f such that the economy is at the boundary between Region P and Region M at $R = 1.015$. We design an algorithm to solve for the infimum of the set of f for which a plant owner stops in a finite number of periods.

Suppose the plant owner stops in T period at a particular value of f. Then $S^t(1; f, w, R)$ as a function of t reaches its peak at T. Define a sequence of f_t such that at $f = f_t$, for $z^* = 1$:

$$S^{t+1}(1; f_t, w, R) = S^t(1; f_t, w, R).$$

Intuitively, f_t tracks the movement in the peak as we vary f. If $f = f_t$, the peak is either t or $t + 1$. As t goes to infinity, the peak shifts to infinity. Because

$$S^{t+1}(1; a, w, r) = U^{t+1}(1; R) + \frac{\lambda^{t+1}}{R^t} q$$

and

$$S^t(1; a, w, r) = U^t(1; R) + \frac{\lambda^t}{R^{t-1}} q,$$

we have

$$f_t = \frac{R^t}{\lambda^t} \left[U^{t+1}(1; R) - U^t(1; R) \right].$$

The calibrated value of f is equal to $\inf_{t=1,2,...} f_t$, which we approximate by $\min_{t=1,2,...,T} f_t$ with T large enough. For any value of f strictly above $\inf_{t=1,2,...} f_t$, there must exist a finite optimal stopping time. For any value of f strictly below $\inf_{t=1,2,...} f_t$, there cannot exist a finite stopping time.

After we calibrate the value of f, we solve for x to target a growth rate of 0.5% at gross interest rate $R = 1.015$.

$$x = \frac{a - f - w}{R - \lambda} + \frac{\beta \Pi}{G - \beta \lambda \Pi} w,$$

where $w = \frac{\lambda \eta}{R - \theta \lambda} a$ and

$$\Pi = \pi^E + \pi^S \frac{\beta R (1 - \pi^E)}{G - \beta R (1 - \pi^S)}.$$