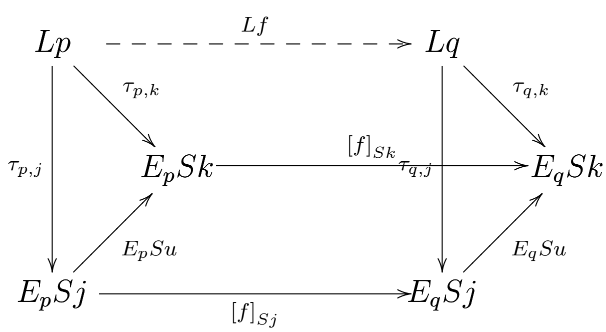
(MAT 313 / PHI 323) Category Theory
Homework and Notes for Chapter 5.

The homework assignment will be listed here.

1 Lecture notes

Theorem 1. *In a functor category, limits can be computed pointwise. More precisely, let* $S : J \longrightarrow X^P$ *be a functor such that each* $E_p \circ S$ *has a limit* (Lp, τ_p) *in* X*. Then there is a unique extension of* L *to a functor from* P *to* X *such that* $p \mapsto \tau_p$ *is a natural transformation* $\tau : \Delta L \Rightarrow S$*, and this* τ *makes* (L, τ) *a limit of* S *in* X^P*.*

Proof. Given $f : p \longrightarrow q$ in P, we need to define an arrow $Lf : Lp \longrightarrow Lq$, and then check that this definition makes L a functor. Recall that $[f] : E_p \Rightarrow E_q$ is a natural transformation. By assumption (Lp, τ_p) is the limit of the functor E_pS. That means that $\tau_p : \Delta(Lp) \Rightarrow E_pS$ is a natural transformation. In particular, for $u : j \longrightarrow k$ in J, we have $E_pSu \circ \tau_{p,j} = \tau_{p,k}$. Thus, the two end triangles in the following diagram commute:



The bottom parallelogram also commutes since $[f]$ is a natural transformation. It follows that the arrows $\alpha_j \equiv [f]_{Sj} \circ \tau_{p,j}$ give a cone from Lp to E_qS. Since Lq is the vertex of a limiting cone on E_qS, there is a unique arrow $Lf : Lp \longrightarrow Lq$ such that $\tau_{q,j} \circ Lf = [f]_{Sj} \circ \tau_{p,j}$ for all objects j of J.

To see that L is a functor, use the uniqueness of the arrow Lf. In particular, 1_{Lp} also gives a morphism of the cone (Lp, τ_p) to itself; and $Lg \circ Lf$ also gives a morphism of the cone $(Lp, [g \circ f] \circ \tau_p)$ to the cone (Lr, τ_r).

We claim now that for each object j of J, $p \mapsto \tau_{p,j}$ is a natural transformation from L to Sj; i.e. it is an arrow in the category X^P. Expanding the definitions of the evaluation functors transforms in the front square of the previous diagram gives

$$\tau_{q,j} \circ L(f) = Sj(f) \circ \tau_{p,j},$$

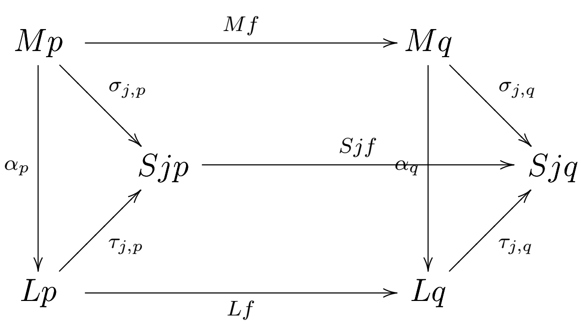
which is precisely naturality of $p \longrightarrow \tau_{p,j}$. In a slight abuse of notation, let $\tau_j : L \Rightarrow Sj$ denote this natural transformation.

Next we claim that $j \mapsto \tau_j$ is a natural transformation from $\Delta(L)$ to S. In other words, for $u : j \longrightarrow k$ in J, we have the following equality of natural transformations:

$$Su \circ \tau_j = \tau_k.$$

Natural transformations are equal just in case they are equal at each coordinate, hence we need to show that $(Su)_p \circ \tau_{p,j} = \tau_{p,k}$ for all p. But $(Su)_p = E_p(Su)$ by definition, and so the equality is just the left triangle of the previous diagram.

Finally we need to show that (L, τ) is the limiting cone on S. To this end, suppose that (M, σ) is another cone on S. For each p, (Lp, τ_p) is the limit of E_pS, and (Mp, σ_p) is a cone on E_pS. Thus, there is a unique arrow $\alpha_p : Mp \longrightarrow Lp$ such that $\tau_{j,p} \circ \alpha_p = \sigma_{j,p}$. To see that α is natural from M to L, consider the following diagram:



The triangles commute by the definition of α_p and α_q. The parallelograms commute since σ and τ are natural. Thus the front square commutes when suffixed by $\tau_{j,q}$. Since this is true for all j, and (Lq, τ_q) is a limiting cone on E_qS, the front square commutes. That is, α is a natural transformation. □

2 The adjoint functor theorem

Lemma 1. *Suppose that $e : v \longrightarrow w$ is an equalizer of the family of all endomorphisms of w. Then any arrow $f : w \longrightarrow v$ is a split epi.*

Proof. ef is an endomorphism of w, hence

$$e(fe) = (ef)e = 1_w e = e1_v.$$

Since e is an equalizer, it is monic; hence $fe = 1_v$ and f is a split epi. □

Lemma 2. *Suppose that $e : v \longrightarrow w$ is an equalizer of all endomorphisms of w. Suppose also that for each object d there is an arrow $g : w \longrightarrow d$. Then each arrow into v is epi.*

Proof. Let $f : d \longrightarrow v$ be an arrow. By assumption there is an arrow $g : w \longrightarrow d$. By the previous lemma, fg is split epi, and hence f is epi. □

Note 1. If the category D has equalizers, then "every arrow into v is an epi" entails that v has *at most* one arrow to any other object.

Theorem 2. *Let D be small complete and locally small. Then the following are equivalent:*

1. *D has an initial object.*
2. *There is a set $\{k_i : i \in I\}$ of objects of D such that for every object d, there is an $i \in I$ and an arrow $f : k_i \longrightarrow d$.*

Proof. (2) $\Rightarrow$ (1) Since D is small complete, $w = \prod_{i \in I} k_i$ exists. Clearly for any object d of D, there is an arrow $g : w \longrightarrow d$. Since D is locally small and small complete, there is an equalizer $e : v \longrightarrow w$ of the set of all endomorphisms of w. Thus, for each object d, there is at least one arrow $ge : v \longrightarrow d$. By the previous lemma, every arrow into v is epi; and since D has equalizers, v has at most one arrow to any object. Therefore v is an initial object. □