(MAT 313 / PHI 323) Category Theory
Homework for Chapter 4. Due on Wednesday, Nov 17.

1. CWM p 86, #1.
2. CWM p 92, #3. There is a typo in the exercise: The adjunction should be $\langle F, G, \varphi \rangle$, that is, F is *left* adjoint to G.
3. CWM p 92, #5.
4. Write the statement of the dual of Theorem 1 on CWM, p 90. Hint: Use the table in Exercise 1, page 92.
5. CWM p 92, #6.

1 Lecture notes

Recall that the Yoneda lemma shows that for $F : D \longrightarrow \mathbf{Set}$ and $r \in D$, there is a bijection $y : \mathrm{Nat}(D(r, -), F) \longrightarrow Fr$. If we replace F for another functor G, or r with another object s, then we get another bijection between Nat($D(s, -)$ and Gs. We now show that the Yoneda bijections form a natural transformation in these two variables.

Proposition 1. *Consider two functors from* $\mathbf{Set}^D \times D$ *to* $\mathbf{Set}$ *given on objects by:*

$$(F, r) \longmapsto Fr, \tag{1}$$
$$(F, r) \longmapsto \mathrm{Nat}(D(r, -), F). \tag{2}$$

The Yoneda correspondence is a natural isomorphism between these two functors.

Before we begin the proof, let us be precise about the definition of these two functors on arrows. In $\mathbf{Set}^D \times D$, and arrow from (F, r) to (G, s) is of the form (μ, f) with $\mu : F \Rightarrow G$ a natural transformation and $f : r \longrightarrow s$ an arrow in D. Functor (1) maps (μ, f) to the diagonal of:

$$\begin{array}{ccc} Fr & \xrightarrow{\mu_r} & Gr \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ Fs & \xrightarrow[\mu_s]{} & Gs \end{array}$$

Functor (2) takes (μ, f) to $\mathrm{Nat}(f^*, \mu)$, which maps arbitrary $\alpha \in \mathrm{Nat}(D(r,-), F)$ to $\mu \cdot \alpha \cdot f^*$.

Proof. We need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Nat}(D(r,-), F) & \xrightarrow{y_{F,r}} & Fr \\
\Big\downarrow{\scriptstyle \mathrm{Nat}(f^*,\mu)} & & \Big\downarrow{\scriptstyle Gf\circ\mu_r = \mu_s \circ Ff} \\
\mathrm{Nat}(D(s,-), G) & \xrightarrow[y_{G,s}]{} & Gs
\end{array}$$

So choose arbitrary $\alpha \in \mathrm{Nat}(D(r,-), F)$. By definition, $y_{F,r}(\alpha) = \alpha_r(1_r)$. Then moving down the right vertical arrow of the diagram we get $\mu_s[Ff(\alpha_r(1_r))]$. Since $\alpha : D(r,-) \Rightarrow F$ is natural, $Ff \circ \alpha_r = \alpha_s \circ f_*$. Thus, $\mu_s[Ff(\alpha_r(1_r))] = \mu_s[\alpha_s(f_*(1_r))] = \mu_s[\alpha_s(f)]$. Clearly the same value results from chasing α counterclockwise around the diagram. Therefore $\{y_{F,r} : (F, r) \in \mathbf{Set}^D \times D\}$ is a natural transformation. □

For future reference, we will call the first diagram in the preceding proof the Yo-Nat diagram.

Proposition 2. *Let $K : D \longrightarrow \mathbf{Set}$. Then K is a colimit of representable functors. In particular, K is a colimit of the diagram*

$$\mathrm{Elts}(K) = (* \downarrow K) \xrightarrow{\Pi} D \xrightarrow{Y} \mathbf{Set}^D$$

where Π is the forgetful functor, and Y is the contravariant Yoneda embedding.

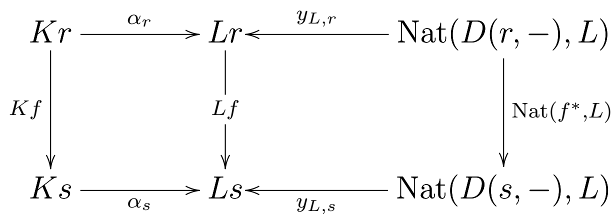
Proof. For $(r, x) \in \mathrm{Elts}(K)$, let $\sigma_{(r,x)} : D(r,-) \Rightarrow K$ be given by $\sigma_{(r,x)} = y^{-1}_{K,r}(x)$. Since y is natural, so is y^{-1}. We claim that $\{\sigma_{(r,x)} : (r, x) \in \mathrm{Elts}(K)\}$ forms a cocone on $Y \circ \Pi$. Indeed, if $f : (r, x) \longrightarrow (s, u)$ is an arrow in $\mathrm{Elts}(K)$, which means that $(Kf)(x) = u$, then

$$\begin{aligned}
\sigma_{(r,x)} \cdot f^* &= \mathrm{Nat}(f^*, 1_K)(y^{-1}_{K,r}(x)) \\
&= y^{-1}_{K,s}((Kf)(x)) \\
&= y^{-1}_{K,s}(u) \\
&= \sigma_{(s,u)},
\end{aligned}$$

where the second equality uses the Yo-Nat diagram with $\mu = 1_K$.

To show that (K, σ) is a limiting cocone, suppose that (L, τ) is an arbitrary cocone on $Y \circ \Pi$. That is, for each $r \in D$ and $x \in Kr$, $\tau_{(r,x)} : D(r, -) \Rightarrow L$ is a natural transformation, and when $f : (r, x) \longrightarrow (s, u)$ is an arrow in $\mathrm{Elts}(K)$ then $\tau_{(s,y)} = \tau_{(r,x)} \cdot f^*$. We need to find a unique morphism $\alpha : K \Rightarrow L$ of cocones. Indeed, for each $r \in D$ and $x \in Kr$, define $\alpha_r(x) = y_{L,r}[\tau_{(r,x)}] \in Lr$.

Check that α is natural: Let $f : r \longrightarrow s$ and consider the diagram

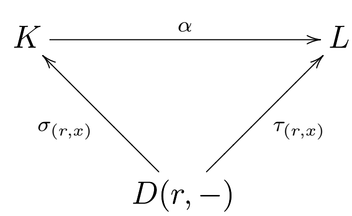


Both squares commute: the left since α is natural, and the right is the Yo-Nat diagram. Then we calculate:

$$\begin{aligned}
(Lf \circ \alpha_r)(x) &= Lf(\alpha_r(x)) \\
&= Lf(y_{L,r}[\tau_{(r,x)}]) \\
&= y_{L,s}(\tau_{(r,x)} \cdot f^*) \\
&= y_{L,s}[\tau_{(s,Kf(x))}] \\
&= \alpha_s(Kf(x)) \\
&= (\alpha_s \circ Kf)(x).
\end{aligned}$$

In the third to last equation, we use that (L, τ) is a cocone and $f : (r, x) \longrightarrow (s, Kf(x))$ is an arrow in $\mathrm{Elts}(K)$.

Check that α is a morphism of cocones, that is the following triangle commutes



Since both $\tau_{(r,x)}$ and $\alpha \cdot \sigma_{(r,x)}$ are natural transformations from $D(r, -)$ to L, it is enough to check that they agree on 1_r in $D(r, r)$. But $\sigma_{(r,x)} = y^{-1}_{K,r}(x)$ means that $\sigma_{(r,x)}$ is induced by x. That is, $[\sigma_{(r,x)}]_r(1_r) = x$. Therefore

$$(\alpha \cdot \sigma_{(r,x)})_r(1_r) = \alpha_r(x),$$

that is $\alpha \cdot \sigma_{(r,x)}$ is induced by $\alpha_r(x)$ and is therefore equal to $\tau_{(r,x)}$.

Finally check that α is the unique morphism of cocones: if $\beta : K \Rightarrow L$ also makes the triangle commute then

$$\tau_{(r,x)} = (\beta \cdot \sigma_{(r,x)})_r(1_r) = \beta_r(x),$$

for all $r \in D$ and $x \in Kr$. Thus, $\beta = \alpha$. □

2 Adjunctions

Proposition 3. *Let $F : D \longrightarrow C$ and $G : C \longrightarrow D$ be functors, and let $\eta : 1_C \Rightarrow FG$ and $\varepsilon : GF \Rightarrow 1_D$ be natural transformations that satisfy the triangle equalities:*

$$(1_F * \varepsilon) \circ (\eta * 1_F) = 1_F \qquad\qquad (\varepsilon * 1_G) \circ (1_G * \eta) = 1_G.$$

Then for each object c of C, the pair $\langle Gc, \eta_c \rangle$ is universal from c to F.

Proof. Suppose that d is an object of D and $f : c \longrightarrow Fd$ an arrow of C. We must show that there is a unique arrow $\hat{f} : Gc \longrightarrow d$ such that $F(\hat{f}) \circ \eta_c = f$. We claim first that $\varepsilon_d \circ Gf$ satisfies this equation. Indeed,

$$\begin{aligned} F(\varepsilon_d \circ Gf) \circ \eta_c &= F\varepsilon_d \circ FGf \circ \eta_c \\ &= F\varepsilon_d \circ \eta_{Fd} \circ f \\ &= 1_{Fd} \circ f \;=\; f, \end{aligned}$$

where we used the naturality of η for the second equation and the first triangle equality for the penultimate equation.

We now show uniqueness: if $Fa \circ \eta_c = f$ then $a = \varepsilon_d \circ Gf$. Indeed,

$$\begin{aligned} \varepsilon_d \circ Gf &= \varepsilon_d \circ G(Fa \circ \eta_c) \\ &= \varepsilon_d \circ GFa \circ G\eta_c \\ &= a \circ \varepsilon_{Gc} \circ G\eta_c \\ &= a \circ 1_{Gc} \;=\; a, \end{aligned}$$

where we used the naturality of ε for the third equation, and the second triangle equality for the penultimate equation.

□

3 Reflective subcategories

Lemma. *Consider an adjunction* $X \underset{G}{\overset{F}{\rightleftarrows}} A$ ($\perp$) *Then for any arrows* $x, y : a \longrightarrow b$ *in* A*, we have* $x \circ \varepsilon_a = y \circ \varepsilon_a$ *iff* $Gx = Gy$*.*

Proof. Since $\varepsilon : FG \Rightarrow 1_A$ is a natural transformation, both squares in the following diagram commute:

$$\begin{array}{ccc} FGa & \xrightarrow{\varepsilon_a} & a \\ {\scriptstyle FGx}\Big\downarrow\Big\downarrow{\scriptstyle FGy} & & {\scriptstyle x}\Big\downarrow\Big\downarrow{\scriptstyle y} \\ FGb & \xrightarrow[\varepsilon_b]{} & b \end{array}$$

Thus $x \circ \varepsilon_a = y \circ \varepsilon_a$ iff $\varepsilon_b \circ F(Gx) = \varepsilon_b \circ F(Gy)$ iff $\varphi^{-1}(Gx) = \varphi^{-1}(Gy)$ iff $Gx = Gy$ (since φ is a bijection). $\square$

Theorem 1. *For an adjunction* $X \underset{G}{\overset{F}{\rightleftarrows}} A$ ($\perp$) *we have:*

1. *G is faithful iff every component ε_a of the counit ε is epi,*

2. *G is full iff every ε_a is a split monic.*

Hence G is full and faithful iff each $\varepsilon_a : FGa \longrightarrow a$ is an isomorphism.

Proof. (1) G is faithful means: for any $x, y : a \longrightarrow b$, $Gx = Gy$ implies $x = y$. All components of the counit ε are epi means: for any $x, y : a \longrightarrow b$, $x \circ \varepsilon_a = y \circ \varepsilon_a$ implies $x = y$. The lemma shows that the antecedent of the two conditionals are equivalent; hence G is faithful iff each component of the counit is epi.

(2) If G is full then there is an arrow $x : a \longrightarrow FGa$ such that $Gx = \eta_{Ga}$. In this case,

$$1_{FGa} = \varepsilon_{FGa} \circ F\eta_{Ga} = \varepsilon_{FGa} \circ FGx = x \circ \varepsilon_a,$$

where the first equality follows from the triangle equalities, and the third by the naturality of ε. Therefore ε_a is a split monic.

Conversely, suppose that ε_a is a split monic, in particular $x \circ \varepsilon_a = 1_{FGa}$. By the previous equation, $\varphi^{-1}(Gx) = 1_{FGa} = \varphi^{-1}(\eta_{Ga})$. Since φ^{-1} is a bijection, $Gx = \eta_{Ga}$. Now let $f : Ga \longrightarrow Gb$. Then

$$G(\varepsilon_b \circ Ff \circ x) = G\varepsilon_b \circ GFf \circ \eta_{Ga} = G\varepsilon_b \circ \eta_{Gb} \circ f = f,$$

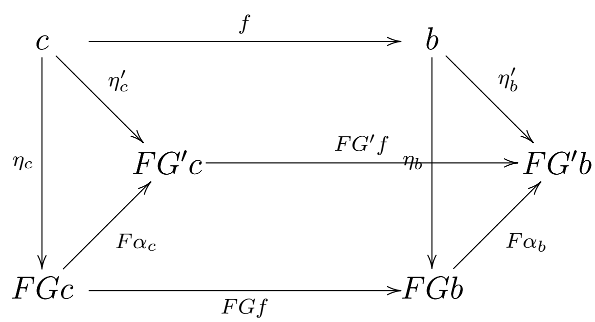
where the final equation uses the triangle equalities. Therefore G is full. □

4 Uniqueness of adjoints

Proposition 4. *Suppose that $G \dashv F$ with unit $\eta : 1 \Rightarrow FG$, and suppose also that $G' \dashv F$ with unit $\eta' : 1 \Rightarrow FG'$. Then there is a unique natural isomorphism $\alpha : G \longrightarrow G'$ such that $\eta' = (1_F * \alpha) \circ \eta$.*

Proof. For an object c of C, the pairs $\langle Gc, \eta_c : c \longrightarrow FGc\rangle$ and $\langle G'c, \eta'_c : c \longrightarrow FG'c\rangle$ are universal arrows from c to F. By the uniqueness of universal arrows, there is an isomorphism $\alpha_c : Gc \longrightarrow G'c$ such that $\eta'_c = F(\alpha_c) \circ \eta_c$. We now show that $(\alpha_c)_{c \in C}$ is a natural transformation.

Consider the following diagram:



The left and right triangles commute, by the definition of α_c and α_b. The front square commutes since η is natural. The top back parallelogram commutes since η' is natural. We can then compute:

$$\begin{aligned} F\alpha_b \circ FGf \circ \eta_c &= F\alpha_b \circ \eta_b \circ f \\ &= \eta'_b \circ f \\ &= FG'f \circ \eta'_c \\ &= FG'f \circ F\alpha_c \circ \eta_c. \end{aligned}$$

Therefore $F(\alpha_b \circ Gf) \circ \eta_c = F(G'f \circ \alpha_c) \circ \eta_c$. Since $\langle Gc, \eta_c \rangle$ is universal from c to F, it follows that $\alpha_b \circ Gf = G'f \circ \alpha_c$. Since this is true for all $f : c \longrightarrow b$, α is a natural transformation.

We already showed that $\eta'_c = F(\alpha_c) \circ \eta_c$, which entails that $\eta' = (1_F * \alpha) \circ \eta$. Finally, if β is a natural transformation that satisfies this equation then

$$F(\alpha_c) \circ \eta_c = F(\beta_c) \circ \eta_c,$$

for all c. Since $\langle Gc, \eta_c \rangle$ is universal, $\beta_c = \alpha_c$ for all c. □

Here is a sketch of an alternate proof of the same result.

Proof. If G and G' are left adjoints of F, then there are bijections

$$D(Gc, d) \xrightarrow{\varphi_{c,d}^{-1}} C(c, Fd) \xrightarrow{\varphi'_{c,d}} D(G'c, d),$$

natural in c and d. In particular, the functors $D(Gc, -)$ and $D(G'c, -)$ are naturally isomorphic. By the Yoneda lemma, Gc and $G'c$ are isomorphic. □