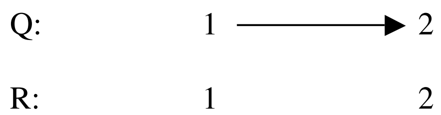
Review Session: Proofs and Counterexamples, Problems 3 and 4

3a. Give a proof that shows that '(x)(y)[Qxy ↔ (z)(Rzx → Rzy)]' implies '(x)Qxx'.

1	(1)	(x)(y)[Qxy ↔ (z)(Rzx → Rzy)]	A
1	(2)	(y)[Qay ↔ (z)(Rza → Rzy)]	1 UE
1	(3)	Qaa ↔ (z)(Rza → Rza)	2 UE
1	(4)	[Qaa → (z)(Rza → Rza)] & [(z)(Rza → Rza) → Qaa]	3 def. ↔
1	(5)	(z)(Rza → Rza) → Qaa	4 &E
6	(6)	Rba	A
∅	(7)	Rba → Rba	6,6 CP
∅	(8)	(z)(Rza → Rza)	7 UI
1	(9)	Qaa	5,8 MPP
1	(10)	(x)Qxx	9 UI

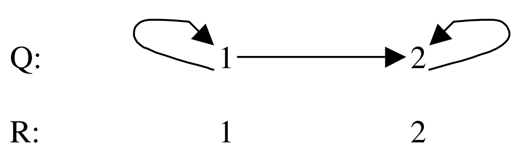
3b. Give an interpretation that shows that ‘$(x)(y)[Qxy \leftrightarrow (z)(Rzx \rightarrow Rzy)]$’ does not imply ‘$(x)(y)(Qxy \rightarrow Qyx)$’.

We need to find an interpretation that makes the first sentence true but the second sentence false. Here is a straightforward way to make the second sentence false:



If this were our entire interpretation, it would make the second sentence false.

However, our interpretation also has to make the first sentence true if it is to show that the first sentence doesn’t imply the second. And, thinking back to the previous problem, we’ve seen that if the first sentence is true, then the sentence ‘$(x)Qxx$’ must also be true. This suggests that the interpretation above won’t work, and that we should consider the following interpretation, which does make ‘$(x)Qxx$’ true:



However, this interpretation doesn’t work either. The problem is that the first sentence is still false relative to this interpretation. To see why, consider the assignment x=2, y=1; under that assignment, the following is false relative to the interpretation above:

$$Qxy \leftrightarrow (z)(Rzx \rightarrow Rzy)$$

(Because, if x=2 and y=1, then ‘Qxy’ is false but ‘$(z)(Rzx \rightarrow Rzy$’ is true.)

But if the sentence ‘$(x)(y)[Qxy \leftrightarrow (z)(Rzx \rightarrow Rzy)]$’ is to be true relative to our interpretation, then the sentence ‘$Qxy \leftrightarrow (z)(Rzx \rightarrow Rzy)$’ has to be true on all assignments of x and y.

Here is an interpretation that resolves this problem:

Q:

R:

DoQ = {1,2}
Ext(Qxy) = {<1,1>, <1,2>, <2,2>}
Ext(Rxy) = {<2,2>}

This interpretation works: it makes the first sentence true, but the second sentence false—and this shows that the first sentence doesn't imply the second.

3c. The sentence '$(x)(y)[Qxy \leftrightarrow (z)(Rzx \rightarrow Rzy)]$' implies one of (i) and (ii) but not the other; give a proof to show the implication in the one case, and give an interpretation to show the lack of implication in the other:

(i) $(\exists y)(x)Rxy \rightarrow (\exists y)(x)Qxy$

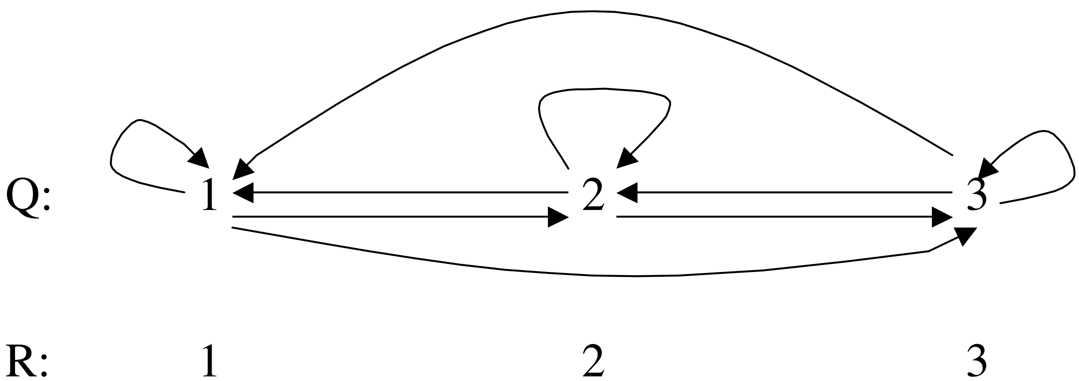
(ii) $(\exists y)(x)Qxy \rightarrow (\exists y)(x)Rxy$

Here is a proof that shows that the sentence implies (i):

Goal: $(\exists y)(x)Qxy$

1	(1)	$(x)(y)[Qxy \leftrightarrow (z)(Rzx \rightarrow Rzy)]$	A	
2	(2)	$(\exists y)(x)Rxy$	A	for CP
3	(3)	$(x)Rxa$	A	for EE
1	(4)	$(y)[Qby \leftrightarrow (z)(Rzb \rightarrow Rzy)]$	1 UE	
1	(5)	$Qba \leftrightarrow (z)(Rzb \rightarrow Rza)]$	4 UE	
1	(6)	$[Qba \rightarrow (z)(Rzb \rightarrow Rza)] \& [(z)(Rzb \rightarrow Rza) \rightarrow Qba]$	5 def. $\leftrightarrow$	
1	(7)	$(z)(Rzb \rightarrow Rza) \rightarrow Qba$	6 &E	
8	(8)	Rcb	A	for CP
3	(9)	Rca	3 UE	
3	(10)	$Rcb \rightarrow Rca$	8,9 CP	
3	(11)	$(z)(Rzb \rightarrow Rza)$	10 UI	
1,3	(12)	Qba	7,11 MPP	
1,3	(13)	$(x)Qxa$	12 UI	
1,3	(14)	$(\exists y)(x)Qxy$	13 EI	
1,2	(15)	$(\exists y)(x)Qxy$	2,3,14 EE	
1	(16)	$(\exists y)(x)Rxy \rightarrow (\exists y)(x)Qxy$	2,15 CP	

Here is an interpretation that shows that ‘*(x)(y)[Qxy* $\leftrightarrow$ *(z)(Rzx* $\rightarrow$ *Rzy)]*’ does *not* imply ‘*(∃y)(x)Qxy* $\rightarrow$ *(∃y)(x)Rxy*’:



DoQ = {1,2,3}
Ext(Qxy) = {<1,1>, <1,2>, <1,3>, <2,1>, <2,2>, <2,3>, <3,1>, <3,2>, <3,3>}
Ext(Rxy) = ∅

Relative to this interpretation, the first sentence is true but the second is false.

4. *Use some reliable method to determine whether or not the following argument is valid. If the argument is invalid, provide a counterexample interpretation. If the argument is valid, explain how you know that there can be no counterexample.*

$$(\exists x)Gx \vee \sim(x)Fx\,,\quad \sim(x)\sim Fx \rightarrow \sim(x)Fx \quad \vdash \quad (\exists x)Fx \rightarrow (\exists x)Gx$$

Let’s begin with a ‘low-tech’ way of trying to find a counterexample.

We want all the premises to be true but the conclusion false relative to our interpretation; thus:

$$\begin{array}{ccccccccccc}
(\exists x)Gx & \vee & \sim(x)Fx\,, & \sim(x)\sim Fx & \rightarrow & \sim(x)Fx & \vdash & (\exists x)Fx & \rightarrow & (\exists x)Gx \\
 & T & & & T & & & & F &
\end{array}$$

If the conclusion is to be false, then ‘(∃x)Fx’ must be true, but ‘(∃x)Gx’ false; this forces us to assign truth values as follows (note that ‘~(x)~Fx’ is equivalent to ‘(∃x)Fx’):

$$\begin{array}{ccccccccccc}
(\exists x)Gx & \vee & \sim(x)Fx\,, & \sim(x)\sim Fx & \rightarrow & \sim(x)Fx & \vdash & (\exists x)Fx & \rightarrow & (\exists x)Gx \\
F & T & & T & T & & & T & F & F
\end{array}$$

Now, given what we have here, ‘~(x)Fx’ must be true if the premises are to both be true; thus:

$$\begin{array}{ccccccccccc}
(\exists x)Gx & \vee & \sim(x)Fx\,, & \sim(x)\sim Fx & \rightarrow & \sim(x)Fx & \vdash & (\exists x)Fx & \rightarrow & (\exists x)Gx \\
F & T & T & T & T & T & & T & F & F
\end{array}$$

And this tells us that we want an interpretation that assigns truth-values as follows:

$v((\exists x)Gx) = F$
$v(\sim(x)Fx) = T$
$v((\exists x)Fx) = T$

So, in our interpretation there can’t be anything in the extension of G, it can’t be the case that everything is in the extension of F, but something must be in the extension of F.

This suggests the following interpretation:

DoQ = {1,2}
Ext(Gx) = ∅
Ext(Fx) = {1}

Relative to this interpretation, all the premises of the argument are true but the conclusion is false, and this shows that the argument is invalid.

Now let's use the small domain method to find a counterexample to the argument. Here is the argument again:

$$(\exists x)Gx \vee \sim(x)Fx\ ,\quad \sim(x)\sim Fx \rightarrow \sim(x)Fx \quad \vdash \quad (\exists x)Fx \rightarrow (\exists x)Gx$$

We want to see whether the conjunction of the premises and the negation of the conclusion is consistent; if it is, then the argument is invalid, since that means that there is an interpretation that makes all the premises true but the conclusion false.

Here is the conjunction of the premises and the negation of the conclusion:

$$[(\exists x)Gx \vee \sim(x)Fx] \ \&\ [\sim(x)\sim Fx \rightarrow \sim(x)Fx] \ \&\ \sim[(\exists x)Fx \rightarrow (\exists x)Gx]$$

Now, let's turn this into a sentence that will be easier to deal with. To do that, we note that '$\sim(x)\sim Fx$' is equivalent to '$(\exists x)Fx$', and that '$\sim(x)Fx$' is equivalent to '$(\exists x)\sim Fx$'. Thus, we get the following:

$$[(\exists x)Gx \vee (\exists x)\sim Fx] \ \&\ [(\exists x)Fx \rightarrow (\exists x)\sim Fx] \ \&\ \sim[(\exists x)Fx \rightarrow (\exists x)Gx]$$

Next, we find a quantifier-free sentence that is equivalent to the one above relative to a domain of one object, and then test that sentence for consistency (see handout). Here is the resulting sentence:

$$[Ga \vee \sim Fa] \ \&\ [Fa \rightarrow \sim Fa] \ \&\ \sim[Fa \rightarrow Ga]$$

This sentence is inconsistent (as a truth-table test reveals).

Given that the equivalent sentence relative to a domain of one object is inconsistent, we next find a quantifier-free sentence that is equivalent relative to a domain of *two* objects, and then test that sentence for consistency. Here is the resulting sentence:

$$[(Ga \vee Gb) \vee (\sim Fa \vee \sim Fb)] \ \&\ [(Fa \vee Fb) \rightarrow (\sim Fa \vee \sim Fb)] \ \&\ \sim[(Fa \vee Fb) \rightarrow (Ga \vee Gb)]$$

This sentence is consistent, as the following truth-assignment (discoverable by truth-tables) shows: v(Ga) = F, v(Gb) = F, v(Fa) = T, v(Fb) = F. And this tells us that the following interpretation is a counterexample to the argument (see handout on Algorithm A):

DoQ = {1,2}
Ext(Gx) = ∅
Ext(Fx) = {1}

And this is the same interpretation we arrived at by the low-tech method.

Finally, let's use Algorithm C to find a counterexample to the argument. Here is the argument again:

$$(\exists x)Gx \vee \sim(x)Fx \,,\quad \sim(x)\sim Fx \rightarrow \sim(x)Fx \quad \vdash \quad (\exists x)Fx \rightarrow (\exists x)Gx$$

Again, we want to see whether the conjunction of the premises and the negation of the conclusion is consistent; if it is, then the argument is invalid, since that means that there is an interpretation that makes all the premises true but the conclusion false.

Here again is the conjunction of the premises and the negation of the conclusion:

$$[(\exists x)Gx \vee \sim(x)Fx] \;\&\; [\sim(x)\sim Fx \rightarrow \sim(x)Fx] \;\&\; \sim[(\exists x)Fx \rightarrow (\exists x)Gx]$$

Now, let's again turn this into a sentence that will be easier to deal with. To do that, we note that '~(x)~Fx' is equivalent to '(∃x)Fx', and that '~(x)Fx' is equivalent to '(∃x)~Fx'. Thus, we get the following:

$$[(\exists x)Gx \vee (\exists x)\sim Fx] \;\&\; [(\exists x)Fx \rightarrow (\exists x)\sim Fx] \;\&\; \sim[(\exists x)Fx \rightarrow (\exists x)Gx]$$

The next step is to put this sentence into 'disjunctive normal form'. (See Lemmon pp. 190-5). You can follow along as I transform this into DNF (no easy task).

$$[(\exists x)Gx \vee (\exists x)\sim Fx] \;\&\; [\sim(\exists x)Fx \vee (\exists x)\sim Fx] \;\&\; [(\exists x)Fx \;\&\; \sim(\exists x)Gx]$$

(In the last step I used the fact that 'A → B' is equivalent to '~A ∨ B', and DeMorgan's Law.)

$$[(\exists x)Gx \vee (\exists x)\sim Fx] \;\&\; \{[(\exists x)Fx \;\&\; \sim(\exists x)Gx \;\&\; \sim(\exists x)Fx] \vee [(\exists x)Fx \;\&\; \sim(\exists x)Gx \;\&\; (\exists x)\sim Fx]\}$$

(In the last step I used the fact that 'A & (B ∨ C)' is equivalent to '(A & B) ∨ (A & C)'.)

$$\{(\exists x)Fx \;\&\; \sim(\exists x)Gx \;\&\; \sim(\exists x)Fx \;\&\; [(\exists x)Gx \vee (\exists x)\sim Fx]\} \vee \{(\exists x)Fx \;\&\; \sim(\exists x)Gx \;\&\; (\exists x)\sim Fx \;\&\; [(\exists x)Gx \vee (\exists x)\sim Fx]\}$$

(In the last step I used the fact that 'A & (B ∨ C)' is equivalent to '(A & B) ∨ (A & C)'.)

$$\{(\exists x)Fx \;\&\; \sim(\exists x)Gx \;\&\; ([\sim(\exists x)Fx \;\&\; (\exists x)Gx] \vee [\sim(\exists x)Fx \;\&\; (\exists x)\sim Fx])\} \vee \{(\exists x)Fx \;\&\; \sim(\exists x)Gx \;\&\; ([(\exists x)\sim Fx \;\&\; (\exists x)Gx] \vee (\exists x)\sim Fx)\}$$

(In the last step I used the fact that 'A & (B ∨ C)' is equivalent to '(A & B) ∨ (A & C)', and the fact that 'A & A' is equivalent to 'A'.)

$$\{(\exists x)Fx \;\&\; ([\sim(\exists x)Gx \;\&\; \sim(\exists x)Fx \;\&\; (\exists x)Gx] \vee [\sim(\exists x)Gx \;\&\; \sim(\exists x)Fx \;\&\; (\exists x)\sim Fx])\} \vee \{(\exists x)Fx \;\&\; ([\sim(\exists x)Gx \;\&\; (\exists x)\sim Fx \;\&\; (\exists x)Gx] \vee [\sim(\exists x)Gx \;\&\; (\exists x)\sim Fx])\}$$

(In the last step I used the fact that 'A & (B ∨ C)' is equivalent to '(A & B) ∨ (A & C)'.)

[(∃x)Fx & ~(∃x)Gx & ~(∃x)Fx & (∃x)Gx] ∨ [(∃x)Fx & ~(∃x)Gx & ~(∃x)Fx & (∃x)~Fx] ∨ [(∃x)Fx & ~(∃x)Gx & (∃x)~Fx & (∃x)Gx] ∨ [(∃x)Fx & ~(∃x)Gx & (∃x)~Fx]

(In the last step I used the fact that 'A & (B ∨ C)' is equivalent to '(A & B) ∨ (A & C)'.)

Now that we have a DNF, the next step is to change each negated simple monadic sentence within each disjunct into an equivalent simple monadic sentence using the quantifier-negation rules so that we can test each disjunct for consistency using Algorithm B. However, I'm only going to do this for the last disjunct, since it is already clear that the other disjuncts are inconsistent, since they contain contradictions. Here then is the relevant sentence that is equivalent to the last disjunct:

(∃x)Fx & (∃x)~Fx & (x)~Gx

Now we use Algorithm B, which in turn tells us to put the following sentence into Algorithm A (see handout):

Fa & ~Fb & ~Ga & ~Gb

This sentence is consistent, as the following truth-assignment (discoverable by truth-tables) shows: v(Fa) = T, v(Fb) = F, v(Ga) = F, v(Gb) = F. And this tells us that the following interpretation is a counterexample to the argument (see handout):

DoQ = {1,2}
Ext(Fx) = {1}
Ext(Gx) = ∅

And this is the same interpretation we arrived at by the low-tech method and the small domain method.