METATHEORY 1

Show: The connective "v" is not by itself truth-functionally complete; that is, there is a truth-functional connective that cannot be expressed using only "v."

Claim: There is no sentence containing only atomic sentence letters and "v"s that is false under every assignment of truth values to its atomic sentences (i.e. that is false in every row of the truth-table). In other words, for every sentence A containing only atomic sentence letters and "v"s, it is not the case that A is false under every assignment of truth values to its atomic sentences (i.e. A is true under some assignment of truth values to its atomic sentences).

Proof (induction):

Base case: A is an atomic sentence.

(To show: A is true under some assignment of truth values to its atomic sentences.)

A is an atomic sentence, so it is contingent, and thus sometimes true. (Another way of putting this step is that since A is composed of only one atomic sentence A, we only have to consider two lines of the truth-table: one where A is true, one where A is false. A is true under one of these assignments).

Inductive Step: A = B v C, where B and C are each true under some assignment of truth values to their atomic sentences (this is the "Inductive Hypothesis").

(To show: A is true under some assignment of truth values to its atomic sentences.)

There is some valuation that makes B true. Since A = B v C is true whenever B is true or C is true, A is true under this valuation.

(Alternative: Reductio. To show: If A is false under every valuation, we get a contradiction with the inductive hypothesis.)

If A is false under every valuation, then, since every assignment of truth values to atomic sentences in A is also an assignment of truth values to atomic sentences in B, and by the truth-table of "v," B must be false under every valuation. Contradiction.

METATHEORY 2

Soundness Theorem for Predicate Calculus: Suppose that there's a correctly written proof in the predicate calculus whose final line is:

D(n) (n) A [some rule]

where D(n) is the set of dependency numbers for line n, and suppose <u>D</u>(n) is the set of sentences written on the lines in D(n). In other words, suppose <u>D</u>(n) $\vdash$ A. Then <u>D</u>(n) semantically implies A; in other words, <u>D</u>(n) $\models$ A.

Completeness Theorem for Predicate Calculus: Suppose that <u>X</u> is a set of sentences of the predicate calculus, and that <u>X</u> semantically implies A. Then there is a correctly written proof in the predicate calculus whose final line is:

D(n) (n) A [some rule]

where D(n) contains the line numbers of all and only those sentences in <u>X</u>.

Soundness of RAA (N.B. this is one "Inductive Step" in the proof of the Soundness Theorem):
(To show: for all applications of RAA, if each of the "input lines" for RAA is good, then the line produced by RAA is good).
Assume the following two lines are good:

i (i) A
D(j) (j) B&-B

In other words, assume that A |= A, and that $\underline{D}$(j) |= B&-B, where $\underline{D}$(j) is the set of sentences written on the lines in D(j). We want to show that the following line is also good:

D(k) (k) -A

where D(k) = D(j) – {i}.
In other words, we want to show that $\underline{D}$(k) |= -A. That is, we want to show that if v is a valuation satisfying $\underline{D}$(k), then v(-A) = T.
Let v be a valuation satisfying $\underline{D}$(k). Since D(k) = D(j) – {i}, we have $\underline{D}$(k) = $\underline{D}$(j) – {A}, so v satisfies $\underline{D}$(j) – {A}. If v(A) = T, then v satisfies $\underline{D}$(j), since $\underline{D}$(j) is a subset of ($\underline{D}$(j) – {A})∪{A}, and both "component" sets are satisfied. But this is impossible: since v satisfies $\underline{D}$(j), v(B&-B) = T (by the "goodness" of line (j)), but this is false for any valuation v. Therefore, v(A) = F. By truth-tables, v(-A) = T.
Therefore, $\underline{D}$(k) |= -A, so line (k) is "good."

METATHEORY 3

A sentence that is a schmautology but not a tautology:

~(∃x)(∃y)(∃z)(Fx & Gx & Fy & -Gy & -Fz & -Gz).

(Intuition: "there are not three objects that have different properties"; the formula in the innermost parentheses is a way of assigning the three objects different properties, thus assuring that they will be different objects). In all domains with at most two elements, we cannot assign extensions of properties in such a way as to make (∃x)(∃y)(∃z)(Fx & Gx & Fy & -Gy & -Fz & -Gz) true. Thus ~(∃x)(∃y)(∃z)(Fx & Gx & Fy & -Gy & -Fz & -Gz) is always true, and hence a schmautology. However, we can think of larger domains with extensions of properties that make ~(∃x)(∃y)(∃z)(Fx & Gx & Fy & -Gy & -Fz & -Gz) false.

Alternatively, we could use particular referents instead of variables:

~(Fm & Gm & Fn & -Gn & -Fo & -Go)

In all domains with at most two elements, we cannot assign extensions of properties *and referents* in such a was as to make (Fm & Gm & Fn & -Gn & -Fo & -Go) true. Thus ~(Fm & Gm & Fn & -Gn & -Fo & -Go) is always true, and hence a schmautology. However, we can think of larger domains with extensions of properties and referents that make ~(Fm & Gm & Fn & -Gn & -Fo & -Go) false.