

VS

maggio-dicembre 1986

**Meaning and Mental
Representations**
edited by Umberto Eco
- Marco Santambrogio - Patrizia Violi

Marco Santambrogio Patrizia Violi	<i>Introduction</i>
Jon Barwise	<i>On the Circumstantial Relation between Meaning and Content</i>
Umberto Eco	<i>On Truth. A Fiction</i>
Gilles Fauconnier	<i>Quantification, Roles and Domains</i>
Ray Jackendoff	<i>Conceptual Semantics</i>
Philip N. Johnson-Laird	<i>How is Meaning Mentally Represented?</i>
George Lakoff	<i>Cognitive Semantics</i>
Wendy G. Lehnert	<i>The Analysis of Nominal Compounds</i>
Roger Schank & Alex Kass	<i>Knowledge Representation in People and Machines</i>
Bas van Fraassen	<i>Identity in Intensional Logic: Subjective Semantics</i>
Yorick Wilks	<i>Reference and its Role in Computational Models of Mental Representations</i>
VS NOTIZIE	<i>Segnalazioni e recensioni</i>

Bompiani

Bas van Fraassen

Identity in Intensional Logic: Subjective Semantics*

After the wonderful success of possible world (truth and reference) semantics in the sixties and seventies, we now gallop off in many directions, to escape its confines. In this paper I shall describe, more or less informally, one of these directions, and the initial findings therein. As focus for our attention I chose *identity*.

Like many others I have been convinced by Kripke and Putnam that there is a distinction between necessity and the *a priori*. In addition I think that logic is not the study of what is necessary, but exactly of what is *a priori*. By “logic” I mean here pure logic, the ground in which can flourish many applied logics – some of which do describe necessity in its various forms and guises. From pure logic I want something different, something more fundamental. Such propositions of identity as “Cicero = Tully”, which are arguably necessary if true but generally not *a priori*, provide therefore a touchstone for the purity of our logic. To arrive at this touchstone, however, we need to travel some ways along my chosen direction – reflecting first of all on the criteria of equivalence in various logics, then on how we can represent reasoning (including *reasoning under suppositions*), and on quantification and abstraction. I commend to you the story of the Zen minds (section 6), offered as the image to replace possible worlds.

I. Distinctions of fact and of reason

1. Modal logic began early in this century with C.I. Lewis’ discontent with a certain feature of then current theories of language. If two sentences have the same truth-value they could be substituted, one for the other, everywhere – similarly for two names with the same referent. We now say that he was discontent because the logic of his day was *extensional*. That means, the criterion of total equivalence was sameness of extension. The term “extension” which means “class of instances”

* Research support by the National Science Foundation is gratefully acknowledged. I also wish to thank Richmond Thomason and Nuel Belnap for helpful discussions. Subjective semantics (the attempt to elucidate logical connections in language through models of reasoning that use the language, rather than models of entities and worlds referred to and described in the language) has taken various forms. I have learned most from the “probabilistic semantics” initiated by William Harper and Hartry Field, and the “epistemic semantics” of Brian Ellis.

when it is applied to a predicate, is here used in a wider sense: the extension of a term is its referent, the extension of a sentence is its truth-value. Thus we have in extensional logic the following three valid inference patterns:

<i>Sentences</i>	<i>Predicates</i>	<i>Terms</i>
$A \equiv B$	$(x) (Fx \equiv Gx)$	$a = b$
$\frac{-A-}{-B-}$	$\frac{-F-}{-G-}$	$\frac{-a-}{-b-}$

I am thinking of logic here as applicable to our natural language, and the interpretation of the logical signs in the above table is supposed to be the usual one in terms of truth value and extension.

Total equivalence amounts to intersubstitutivity everywhere, and I'll call a logic *extensional* if it has the above three criteria for total equivalence. For example, in an extensional logic, the criterion for total equivalence of two predicates F and G is that $(x) (Fx \equiv Gx)$ be true. With respect to the reasoning depicted ("governed") by such a logic this means: if $(x) (Fx \equiv Gx)$ be assumed (to be true), then F and G may be systematically substituted for each other everywhere.

2. A logic is generally called *intensional* if it is not extensional, but let us distinguish at once between *intensional in its narrow sense*, or as I shall also say *modal*, and other sorts of non-extensional logic. In a modal logic there will be a necessity sign, used to formulate its criteria of total equivalence:

$\Box(A \equiv B)$	$\Box(x) (Fx \equiv Gx)$	$\Box(a = b)$
$\frac{-\Box A-}{-\Box B-}$	$\frac{-\Box F-}{-\Box G-}$	$\frac{-\Box a-}{-\Box b-}$

In the usual semantic account we now say that between necessary equivalents, there is no difference in any possible world. The Medievals said: even God could not create a case of the one which is not a case of the other.

I do not intend to deride possible world semantics, which I think is fine as far as it goes. Its way of looking at things does anyway not date from the fifties, but has for example characterized Anglo-Saxon philosophy for a century. Examples may be found in William James' essay "The Dilemma of Determinism" (1884: especially his treatment of the example of a choice of walking home by either Divinity Ave or Oxford Street) and Bertrand Russell's *The Problems of Philosophy* (1912: see especially in Ch. VII, his discussion of $2 + 2 = 4$ and all men being mortal).

But I do think that, just as extensional criteria of total equivalence wipe out important distinctions, so do the modal criteria.

3. Turning now to non-modal non-extensional logic, there is no uniform nomenclature. We hear “hyper-intensional logic” and “fine-grained distinctions”, but mostly there are names for specific such logics, all of which are regarded with suspicion by almost everyone besides the adepts.

The Medievals discussed distinctions of fact (*in re*) but also distinctions of reason. Among the latter the simplest was the one where there is a possibility of a distinction of fact – the case in which God could have, though actually did not, create a case of the one which was not a case of the other. Such is the distinction between rational animal and featherless biped. But there are further distinctions, distinctions of reason where even God could not tear asunder.

Duns Scotus is especially known for the *formal distinction* between transcendentals, i.e. between properties which everything must have. In terms equally reminiscent of Plato’s *Sophist*, we can mention the transcendentals of being and unity: everything that is, is one – and everything that is one, is – and no exceptions could exist; yet there is a distinction. We should also mention especially Francis Suarez, “On the Various Kinds of Distinction”, *Disputationes Metaphysicae*, Disp. VII.¹ But in our century, too, and indeed recently, such distinctions have been drawn.

4. I shall refer, for now, to only one case. The discussion of proper names by, for example, Russell, Ruth Marcus, and Kripke, led to the latter’s theory of rigid designators and the well-known view that “Cicero = Tully” is necessary if it is true at all. Hilary Putnam added a corresponding view for common nouns: “water = H₂O” is necessary if true at all. But for neither example would it be plausible to say that it could be known *a priori*. In a strict sense, their truth is independent of the way the world is; yet no amount of reasoning could have led to them as conclusions.

It is sometimes argued that this distinction is already observed in modal logic. Only the sentence “Cicero = Cicero”, but not “Cicero = Tully” is a *theorem* of modal logic (sound and complete under the standard interpretation) although the proposition expressed is necessary in both cases. But that is not sufficient to show that the relevant distinctions are genuinely observed. For the most general necessity operator \Box – the one that literally corresponds to truth in all possible worlds – we can define

“ $A \rightarrow B$ ” for “ $\Box (A \supset B)$ ”

“ $A \equiv B$ ” for “ $(A \rightarrow B) \ \& \ (B \rightarrow A)$ ”

and then $A \equiv B$ entails the substitutivity of A for B everywhere – their total equivalence. But in these terms we have the theorems

$$\begin{aligned} (\text{Cicero} = \text{Cicero}) \rightarrow (\text{Cicero} = \text{Tully}) \ . \ \text{or} \ . \\ (\text{Cicero} = \text{Tully}) \rightarrow (\text{Cicero} \neq \text{Cicero}) \end{aligned}$$

¹ Tr. C. Vollert, Marquette University Press, 1947.

and

$$\begin{aligned} (\text{Cicero} = \text{Tully}) &\equiv (\text{Cicero} = \text{Cicero}) \text{ . or .} \\ (\text{Cicero} = \text{Tully}) &\equiv (\text{Cicero} \neq \text{Cicero}) \end{aligned}$$

These theorems express the fact, build into logic, that an identity statement must have the same status as *either* a tautology *or* a self-contradiction. Unlike other statements, its status (although we may not know it) cannot be anywhere between these two extremes. The case is exactly analogous to the retort a disciple of Quine could have given to the call for modal logic: “Cicero = Cicero” may be materially equivalent to “Socrates is Greek”, but that is not a theorem. Indeed; but it is a theorem of extensional logic that

$$\begin{aligned} (\text{Cicero} = \text{Cicero}) &\supset (\text{Socrates is Greek}) \text{ . or .} \\ (\text{Socrates is Greek}) &\supset (\text{Cicero} \neq \text{Cicero}) \end{aligned}$$

$$\begin{aligned} (\text{Socrates is Greek}) &\equiv (\text{Cicero} = \text{Cicero}) \text{ . or .} \\ (\text{Socrates is Greek}) &\equiv (\text{Cicero} \neq \text{Cicero}) \end{aligned}$$

and with such theorems, involving the strictest expressible equivalence proper to that logic, the language cannot accommodate the sought-for distinctions.

I urge therefore the need for a “purer” logic – one in which “Cicero = Tully” can have, in any given model, a status different from the *a priori* truth and also from the *a priori* falsehood. This must be urged for the same reasons, *mutatis mutandis*, as we urge against extensional logic that “Socrates is Greek” should not be constrained to have *either* the status of “Cicero = Cicero” *or* that of “Cicero ≠ Cicero”.

II. Subjective Semantics: Minds and Propositions

5. So I am going to talk about pure logic, the theory of the *a priori* and I mean to do so still from a semantic point of view. Now I don’t want to say anything against the usual semantics of truth and reference, but for the present purpose it is obviously not far-reaching enough. It may perhaps be supplemented by something more that is sometimes called *subjective* or even solipsistic semantics, because it does not proceed in terms of what does or even could exist. I profess no imperialism on its behalf. On the other hand, I realize that even if (or rather, especially if) it can be successfully developed, there will be special problems of coexistence for the two sorts of semantics. But reduction of one to the other, or replacement by one for the other, like any sort of imperialistic solution, is probably just the wrong thing to try.

6. I really think that the proposition (by which I mean, the semantic value to be associated with a sentence) is the basic topic of concern in semantics. But I don’t want to start with propositions, because that would make them seem like some Platonist realm of strange entities.

In possible-worlds-truth-and-reference semantics (let's say, standard semantics), propositions are constructs: they are sets of possible worlds. *First* you say: a proposition is true or false in each world. *Then* you add, we may as well *identify* a proposition with the class of worlds in which it is true. You might as well, because you are anyway unable to draw distinctions between propositions which are true in exactly the same worlds.

So let me try this: there are minds, conceivable minds, and they *suppose* some propositions. By this I mean they treat them as "given" or regard them so, or suppose them to be true, as basic premises for all reasoning. And now I'll add: we might as well identify a proposition with the set of minds in which it is supposed, or as I shall also say, all minds in which it is *held* (to be true).

Of course, you may think that this is only to replace one metaphysical metaphor by another, and will result only in a verbal difference. But there is one major difference in that a mind may easily suppose neither a given proposition, nor any incompatible with it. In addition, I hasten to point out that there are obvious relations among minds which have no counterparts on worlds. First of all, some minds are more dogmatic, or *stricter* than others:

$$x \leq y, \text{ } x \text{ is at least as strict as } y, \text{ exactly if} \\ x \text{ supposes all that } y \text{ supposes.}$$

As gloss and symbol suggest, this is a partial ordering. At the very bottom lies the strictest mind of all, which holds all propositions as suppositions. Ever increasing assumptions or dogmatism on more and more subjects has driven it to insanity – it is called \emptyset , the abnormality, the *memento mori* of dogmatic thought. As you go up in the structure, you find freer and freer minds; at the top are *Zen minds* which are subordinate to none but the other Zen minds, holding or supposing only what all minds suppose.

7. We must widen this relationship among minds. Let us say that one mind x is *subordinate* to a set W of minds (briefly, $x \text{ Sub } W$) exactly if x holds (supposes) whatever all members of W hold.² This must be a primitive relation in our construction; the *preceding* \leq is a special case (x is at least as strict as y exactly if x is subordinate to the set $\{y\}$). But it must be a relation such that we will be able to say: x is subordinate to W exactly if x is a member of every proposition which includes W as a subset. It follows then that subordination needs to have the following properties:

- (a) If x is a member of W then x is subordinate to W

² This notion of subordination is analogous to the construal of superposition in quantum logic, which suggested some of the constructions that follow. For reasons to think of quantum logic, like intuitionistic logic, as belonging to the family of modal logics, see Dalla Chiara.

- (b) if all members of U are subordinate to W , and x is subordinate to U , then x is also subordinate to W .
- (c) the abnormal mind is subordinate to every set W .

If W is a set of minds let us call the *span* of W exactly the set of minds that are subordinate to W – and denote this span as $[W]$. It follows now from the above properties that $[...]$ is a so-called closure operation. Also since a proposition is to be identified with the set of minds which hold (suppose) it, a proposition must contain all the minds subordinate to it. Hence we have now a precise way to capture this proposal concerning the propositions.

a *proposition* is a closed set of minds; that is, a set W such that $W = [W]$.

We also have a precise way to characterize the Zen minds, at the top of our hierarchy:

x is a Zen mind exactly if it is subordinate to a set W only if $[W]$ contains all minds.

i.e. only if $[W] = K$, the set of all minds. We may also call $[\{x\}]$ the span of the mind x ; then this becomes

x is a Zen mind exactly if its span is K , the set of all minds.

These minds hold or suppose only what all minds hold, and are free from peculiar beliefs of their own; they are completely undogmatic. (The span of a mind is, as it were, the conjunction of all the propositions it holds.)

In classical logic, also classical modal logic, the propositions always form a Boolean algebra. This need not be so here, and indeed, pure logic should not be so parochial. We have the result:

Theorem The propositions form a complete lattice of sets, with $\Phi = \{\emptyset\}$ as minimum, K as maximum, partially ordered by the relation of set inclusion, set intersection as *meet* (“conjunction”) and the operation $\cup X = [UX]$ as *join* (“disjunction”).

This follows at once from elementary results about closure operations. Almost every logic known to mankind allows of a full set of models in which the propositions form this sort of structure. This includes most certainly classical, intuitionistic, quantum, and relevance logics.

8. I have been saying “hold” and “suppose” as if they were interchangeable. Really they are not. To hold a belief is the most definite, positive epistemic attitude one can have. To suppose something, on the other hand, is not even to hold it momentarily, but to hold it momentarily with a mental reservation, “for the sake of argument”. Yet this

is a very important act, for reasoning would be completely impotent if it could not include reasoning under suppositions. This is what we must now explain.

Let the span of the mind x be briefly denoted as $[x]$. The mind x holds a proposition P exactly if $[x]$ is part of P . We can imagine this mind x adding one more proposition, say A , to what it holds. Then it becomes a different mind, whose span is equal to $[x] \cap A$. Let us therefore call this different mind $x^A - x$ *conditioned on A*. But instead of changing in this way, the mind x could merely imagine itself doing so, and ask itself what it would hold if it added A to all it does hold. This is reasoning under the supposition that A . If it finds that this other mind x^A holds B , then it can say of itself that it holds B *on the supposition that A*, or holds the proposition $(B \text{ if } A)$.

It appears therefore that we can introduce a *conditional* or *implication operation* as follows:

x is a member of $(A \rightarrow B)$ exactly if $x^A \in B$.

But now we face several questions. First, if x is a mind, and A a proposition, is there also such a mind as x^A - that is, another mind whose span equals $[x] \cap A$? And secondly is this set $(A \rightarrow B)$ really a proposition - that is, is it a closed set? The answer to both questions will in general be *no*. If it is *no*, we are dealing with a model of minds which are not capable of reasoning under suppositions. Of course, it is of great interest to look at the sorts of mind which are capable of this.

9. By a *frame of minds* I shall mean a set K (call the members *minds*) with a special member \emptyset and a special relation *Sub* (subordination) such that (a) - (c) in section 7 are satisfied. Now I want to look at the sort of frames in which the minds are capable of conditional reasoning. Let us first define a kind of conditional subordination. This is a defined and not a primitive relationship:

x is *A-subordinate* to W exactly if
for all propositions C : if $y^A \in C$ for each y in W , then $x^A \in C$.

Thus subordination *simpliciter* is the same as K -subordination, because $y^K = y$ always. Let us now call *I-frame* any frame for which the following postulate is satisfied:

Postulate If x is a mind and A a proposition, then x^A is a mind too, and moreover, subordination always implies A -subordination.

This last clause means: if x is subordinate to W , then also x is A -subordinate to W .

Now we have a very nice theorem: the logic by which these minds reason, is *intuitionistic logic*. That is, the I -frames are really models of

that logic, and intuitionistic logic is complete with respect to these models. (See the appendix for a proof-sketch.)

I called this a very nice theorem, but perhaps you were dismayed – must pure logic lead so quickly to intuitionistic logic? What if that is not your favorite? Well, there are other sorts of frames of minds, which you may prefer.

Let us consider a relation different from subordination, namely *radical disagreement* or *orthogonality*. Such a disagreement must be experienced when one mind holds a proposition which another rejects. We have as yet no notion of rejection, but we can see intuitively that this relation of disagreement, to be written as \perp , must have the following properties:

- $x \perp x$ if and only if x is the abnormal mind,
- $x \perp y$ only if $y \perp x$,
- $x \perp \emptyset$ for every mind x .

Such a relation I shall call an *orthogonality* relation on the frame. For the one that represent disagreement, there must be a connection with subordination. Suppose that x is subordinate to W , and z is orthogonal to every member of W . Since x holds whatever the members of W hold in common, z must also be in disagreement with x . Thus:

- (a) x is subordinate to W only if $z \perp W$ implies $z \perp x$, where $z \perp W$ means that $z \perp y$ for each member y of W . (To say that z *rejects* proposition P evidently means that $z \perp P$.)

Conversely, let us suppose that any z which is orthogonal to W is also orthogonal to x . If that orthogonality is radical disagreement, should we conclude that x is subordinate to W ? Well suppose x is not subordinate to W . Then there is some proposition A held by all members of W but not by x . It is possible to imagine then a mind z which holds all that x holds, and holds nothing else, but rejects A . (This could be x itself!) In that case z would be orthogonal to all members of W but not to x . Thus we conclude also:

- (b) x is subordinate to W if $z \perp W$ implies $z \perp x$.

This is enough to go on; let us call an *O-frame* one for which there exists an orthogonality relation \perp such that (a) and (b) hold. In O-frames, closure has a second meaning:

- (c) $[W]$ = the set of minds such that for all minds y , if $y \perp W$ then $y \perp x$.

We have again a very nice theorem: the minds in an O-frame of minds reason in accordance with orthologic (“baby quantum logic” as it is sometimes called). That is, the O-frames are in effect models of orthologic, and orthologic is complete with respect to this set of models. (Again, see the Appendix for a proof sketch.)

But you may still be dismayed, if you are of a more classical turn of mind. You may then prefer C-frames, n. l. the sort of frame which is at once an I-frame and a O-frame. *Classical logic* is sound and complete with respect to the C-frames of mind. So there is room for all the usual logical persuasion. It would be nice to have a further story of this sort to accommodate relevance logics. The new result of Alasdair Urquhardt makes one conjecture wistfully that this area, always closely associated with lattices, can also be brought into the present fold; but I don't know.

III. Quantification and Identity: Transformation Semantics³

10. In semantics, you describe models for language. In models of subjective semantics we could have some familiar denizens of truth and reference semantics, such as propositions (though differently conceived). But it would make no sense to try and introduce counterparts to most of the familiar entities to be found there. For example, I think it would be a mistake to introduce some counterpart of a domain of discourse, such as a mental picture gallery or set of individual concepts, or whatever, to function as surrogate referents. Some such things may eventually appear, but they can't play a basic role. I have two reasons for saying this.

The first comes from thinking about pictures. A picture of Socrates has propositional content; it shows Socrates in repose or excited, sitting or running or flying. So a representation can't be like a bare referent; and indeed there must be many pictures of any one thing, and they never just represent it. So they cannot be good surrogate referents, since we really refer to only one thing by means of many of them, and may only *refer* to it, without describing it.

The second reason comes from thinking about reference without representation. I could even speak about there being things which never have been and never will be pictured, described, or conceived individually. So the surrogates would never be enough anyway.

No, if we are going to do subjective semantics, we must eschew not only reference but also all surrogate reference. Representation there may be, but surrogate reference it is not.

What this means is that I shall make up everything from propositions and operations on propositions. But beforehand I'll motivate it in two ways. The first way is to explain how we can use syntactic analogues, while keeping their referential interpretation "bracketed". The second is to sketch a continuation of our new metaphysical metaphor of conceivable minds engaged in supposition.

³ This section contains an informal exposition of my "Quantification as an act of mind" (briefly, QUAAM); the ideas are not entirely unchanged. Specifically, I think I now understand better what it means to suppose a general proposition.

There is a certain Correspondence Principle that has always guided semantics: the semantic value of a complex expression $F(E_1, \dots, E_n)$ is a function $\Psi(|E_1|, \dots, |E_n|)$ of the semantic values $|E_1|, \dots, |E_n|$ of its parts. This *suggests* but *does not imply* that a proposition is a complex entity of the sort Russell envisaged, built up from parts which are referred to by parts of the sentence that express the proposition. A set of possible worlds has no such parts, neither does a closed set of minds. But once you observe this, you may be able to let the Correspondence Principle guide you to other correspondences. The significant correspondences, I shall submit, are between operations or transformations acting on sentences and acting on propositions. For example, Fb comes from Fa by the substitution of b for a . Well, the proposition $|Fb|$ comes from $|Fa|$ by a corresponding operation, which I shall posit.

Of course it is no use just positing operations any more than propositions themselves – we need a metaphysical metaphor to light our way. So let us see what a mind can suppose.

Imagine I ask you to suppose that Tom Schneider is rich, happy, handsome, a bit small for his age but rather paunchy, You have no idea who Tom Schneider is. You do have the idea that the world has a limited number of inhabitants, and if you did identify Tom, it would be as one of them, but you have made no particular identification. All this is plausible, is it not? But now, *what exactly are you supposing?*

Before answering that, consider the ordinary language of real number algebra, with such sentences in it as

- (1) $2 + 3 = 6$
- (2) $a + b = c$
- (3) $(x)(x + x + a = 2x + a)$

I could ask you to suppose (1), as initial premise of a *reductio*; if you do, you become \emptyset . Can I ask you to suppose that $a + b = c$? Certainly; we can use that as a premise and begin a deduction. If you suppose that $a + b = c$ you are *not* treating this as implicitly quantified – you are supposing *neither* that $(x)(y)(z)(x + y = z)$ *nor* merely that $\exists x \exists y \exists z (x + y = z)$. But neither is there a specific numerical identity, like (1), that you are supposing. Only if you *had* identified a as 2, b as 3, c as 6, would you be supposing (1). Well, what exactly are you supposing?

I'll offer an answer to such questions. You are supposing a proposition, but it is a proposition which depends on certain parameters a, b, c , which you may or may not identify as specific entities – you may or may not “fix their values”. This proposition is a sort of generic proposition, “general” in the sense, derided by Berkeley, of Locke's general triangle.

Syntactically, the sentence $a + b = c$ can be transformed by substitution in two ways: into $a + a = c$ or into $a + 2 = c$. The first substitutes a parametric term a for b ; the second substitutes a numerical constant for

b. I will introduce two corresponding operations on propositions: *variation* and *instantiation* (or *fixing*).

At some point we should ask the question (answered for the dyadic operation \rightarrow in terms of the operation that transforms mind x into x^A) what operations or relations between minds can engender these operations on propositions. At the moment I do not have a satisfactory theory about that, so let us look into the abstract theory.

11. Let us talk about propositions again. Before looking at them from close up, let me list the features of a collection of propositions that could make up a model.

- I. The proposition form a complete lattice, with top element K (the *a priori*), bottom element Φ (the *absurd*), partial ordering \leq (implication relation), conjunction (meet) \wedge , disjunction (join) \vee , ("complete" in that every set of propositions has a conjunction and disjunction).

Any such lattice will do, for the time being; and this assumption is so minimal that no logic I know is disqualified by it. But I will add one feature that does limit the subject a bit.

- II. There is an "implication" operator \rightarrow ("ply"), such that $A \rightarrow B = K$ if and only if $A \leq B$

Quantum logic and relevance logic are the usual exceptions to assumptions about conditionals, but they are not ruled out by the assumption that all models may have *this* feature. So the field is still very wide. We can define our strongest equivalence here by

$$"A \leftrightarrow B" \text{ for } "(A \rightarrow B) \wedge (B \rightarrow A)"$$

We notice that $(A \leftrightarrow B) = K$ if and only if $A = B$, so that condition certainly guarantees substitutivity everywhere (when the syntax is interpreted in such a model).

12. Now I want to talk about quantification, and I shall use syntax as a *pons asinorum*. From the sentence Fa we can make up the term $\hat{a}Fa$, and from Fb , the term $\hat{b}Fb$ — these two terms stand for the properties that a and b must have, respectively, for Fa and Fb to be true. But of course, these must be the same property. The sentence $(x)Fx$ says that this property is universal, everything has it, so it can be written as $U\hat{a}Fa$.

We can look upon this situation in two ways. The first is to think of a proposition which is "generic" like Locke's general triangle, something like

$$A = (\text{a certain entity is } F)$$

and then imagine this generic proposition being made definite into one in which the subject is "fixed":

$$A_1 = \textit{this} \text{ entity is } F$$

$$A_2 = \alpha \text{ is } F$$

$$A_3 = \beta \text{ is } F$$

and so forth. The second way we can look at it is by imagining that some

“specific” proposition is changed by altering its subject:

$$\begin{aligned} B &= \text{this entity is } F \\ B' &= \text{that entity is } F \\ B'' &= \alpha \text{ is } F \\ B''' &= \beta \text{ is } F \end{aligned}$$

and so forth. The two processes are different: A is “made more specific” while B is altered in a way that leaves it “equally specific”. But each of the processes we have imagined is a very general one. They are:

Take this parameter and fix its value at - - - - -

Take this fixed value and alter it to - - - - -

Each of these processes is a *transformation* of all propositions at once (though it does not affect all equally). The transformation preserves structure; if it be called f we have

$$\begin{aligned} f(A \wedge B) &= f(A) \wedge f(B) & f(\bigwedge_i A_i) &= \bigwedge_i f(A_i) \\ f(A \vee B) &= f(A) \vee f(B) & f(\bigvee_i A_i) &= \bigvee_i f(A_i) \\ f(K) &= K \end{aligned}$$

When these equations hold, and only then, will I call the function a transformation. If it additionally has the nice feature

$$f(A \rightarrow B) = f(A) \rightarrow f(B)$$

I’ll call it *normal*.

13. We have found two sorts of transformations that are somehow associated with abstraction and quantification. Should we now choose one to start with? Actually, we can discover quite a bit about these two topics without making any choice, but just thinking about transformations generally. Let us define:

An *abstractor* is a set of transformations.

If b is an abstractor, the *b*-*abstract* of proposition A is the set

$$bA = \{gA : g \in b\}.$$

If $bA = \{A\}$ we call b *irrelevant* to A , or say that A does not depend on b .

I write “ gA ” as short for “ $g(A)$ ”, the proposition that A is turned into by g . In terms of the syntactic *pons asinorum*, $\hat{a}Fa$ is being associated with the set of all propositions that Fa can be turned into, so as to keep everything unaffected by abstraction from a : Fa, Fb, Fc, \dots etc. (But the syntax may give out here; perhaps not all the relevant propositions are expressed in the language.)

The *universal quantifier* \forall_b associated with abstractor b is defined by

$$\forall_b A = \bigwedge (bA)$$

The abstract bA is a set; $\forall_b A$ is the meet of that set. (Note that quantifiers are here not pieces of language but operations in the models for language.)

Theorem Let b be any abstractor and let \forall be its associated quantifier.

Then:

- (a) $\forall K = K$
- (b) if $A \leq B$ then $\forall A \leq \forall B$
- (c) if $\wedge X \leq B$ then $\wedge \{\forall A : A \in X\} \leq \forall B$
- (d) $\forall A \leq gA$ if $g \in b$
- (e) $\forall A \leq \forall \forall A$ if, for any $g, g' \in b$, the proposition $gg'A$ is also in bA .

Let us just look at what all this means.

(a) This corresponds to the rule of simple universal generalization: if A is *a priori*, so is $\forall A$.

(b) This corresponds to conditional reasoning with the quantifier (generalized universal generalization): if A implies B , then $\forall A$ implies $\forall B$.

(c) This generalizes the rule still farther, to infinite sets of premises.

(d) This corresponds closely to universal instantiation. Suppose $(x)Fx$ is written $U\hat{a}Fa$, and $F1$ is a relevant variant of Fa ; then $U\hat{a}Fa$ implies $F1$.

(e) Here we see the principle of vacuous quantification; the proviso holds specifically in two interesting cases:

(e1) b is *destructive*: if $B \in bA$ and $g \in b$ then $gB = B$

(e2) b is a *semigroup*: if $g, g' \in b$ then so is gg' .

Clearly if you fix the value of a parameter at a certain point, and then you try to fix it again, nothing happens because it has changed from fixable to fixed already. (The syntactic analogue is the transformation of "a man is happy" or "x is happy" into "Peter is happy".) Hence at least the first sort of transformation we looked at is destructive.

The importance of the theorem is this: if you think of universal quantification as the simultaneous assertion of all results of a set of transformations, you immediately get the right basic properties — plus in addition a framework for introducing new assumptions (in terms of what the abstractors are like) which may lead to interesting sorts of models.

14. I said that terms $\hat{a}Fa$ and $\hat{b}Fb$ must stand for the same property, sentences $U\hat{a}Fa$ and $U\hat{b}Fb$ for the same propositions. So there must be some relation between the two sorts of expressions that allows this to happen.

You will have noticed that I think of abstractors in a curious, "active" way, as sets of transformations. One intuition is that of fixing a value; the generic proposition A , which depends on some parameter, is transformed into another, by fixing a value for that parameter. All the transformations being thought about there, do the same sort of job — fixing the value of *that* parameter. This close association allows us to identify the parameter with the abstractor whose members fix the value of that parameter. Think of it this way:

$$\begin{array}{ll} b = h(b,1), h(b,2), \dots & h(x,i) \text{ fixes the value of} \\ a = h(a,1), h(a,2), \dots & \text{parameter } x \text{ at } i \end{array}$$

This looks a little circular; but really it has the unobjectionable form of “the Jones family = {The father of the Jones family, the mother of the Jones family,...}”. Now we can alter a proposition which depends on parameter b , by changing its dependence to parameter a ; we can also think of fixing the values of parameter a and b in the same way. At this point drop the picture, toss away the ladder; but you must keep the idea that between the members of abstractors a and b there can be a systematic correlation.

The easiest way to do this is to think of a set G^* of U_T -functions whose members take abstractors into members of those abstractors:

if a is an abstractor and $g \in G^*$ then $g^a \in a$;
 moreover, $a = \{g^a : g \in G^*\}$.

In addition, the transformation that changes a proposition dependent on b into one with exactly similar dependence on a , may be called (a-b) – read it as “a for b” – and then we must have:

The functions:

$g^a g^b$
 $g^a(a-b)$
 $g^b(b-a)$

are all the same function.

This is again easiest to understand in analogy with the syntax. Let $s(x,i)$ replace the term x by the numeral i , and let (x/y) replace all occurrences of y by x :

$$\begin{aligned} s(a,1)s(b,1)Rab &= R11 \\ s(a,1)(a/b)Rab &= s(a,1)Raa = R11 \\ s(b,1)(b/a)Rab &= s(b,1)Rbb = R11 \end{aligned}$$

If our abstractors are correlated in this fashion we can deduce:

Theorem If $A = (a-b)B$ and $B = (b-a)A$,
 then $aA = bB$ and $\forall_a A = \forall_b B$.

The antecedent says that A and B depend on a and b respectively in exactly the same way.

From now on I shall assume we have a correlated set of abstractors; their members g^a I shall call *instantiations* (of a), and the transformations (a-b) I shall call *variations*. These are my precise mathematical correlates of the two processes I described intuitively above.

Obviously the properties of these abstractors and variations between them need to be spelled out further than I have here. I will just assert, without proof, that a few minimal assumptions suffice to yield all required properties. Their theory has a certain completeness, in that if the abstractors are *normal* (i.e. contain only normal transformations) then all functions defined by composing instantiations and variations are strictly calculable.

15. I turn now to identity. To begin, I propose to regard a name as the label of an abstractor. Suppose “Cicero” is a name, and “Cicero is hap-

py” is a sentence that stands for the proposition A. The abstractor a which “Cicero” stands for produces the abstract denoted by “Ĉicero (Cicero is happy)” which is also denoted by “Ĉato (Cato is happy)”. Thus aA contains all the propositions that could be expressed by sentences of form “– is happy”. (Note: “could be”, not merely “are”.)

Now what proposition does “Cicero = Tully” stand for?

Well, what happens if I replace “Cicero” by “Tully” in this sentence? It becomes a tautology. Similarly for the sentence “If Cicero is rich then Tully is rich”, and for many others, like “Either Cicero is rich or Tully is not”. That is really important in this case, because the sentence “Cicero = Tully” when introduced as a premise, sanctions exactly that substitution transformation.

On the side of propositions, we have the case of a transformation into the *a priori* proposition, K. Define, therefore, for a transformation f:

$C(f) = \{A : fA = K\}$ – the *core* of f

$I(f) = \bigwedge C(f) = \bigwedge \{A : fA = K\}$ – the *identity proposition* of f.

How much can we deduce here if we know very little about the transformation? Let us assume that f is idem-potent ($ffA = fA$) and that it is normal. Both assumptions hold already for our variations as well as instantiations.

- Theorem*
- (a) $I(f) \leq A$ iff $f(A) = K$
 - (b) $I(f) \leq (A \rightarrow fA)$
 $I(f) \leq (fA \rightarrow A)$
 - (c) provided $K \rightarrow A = A$ for all A;
 $I(f) = \bigwedge \{A \leftrightarrow fA : A \text{ a proposition}\}$
 - (d) provided $A \wedge (A \rightarrow B) \leq B$ for all A,B:
 $I(f) \wedge A \leq fA$
 $I(f) \wedge fA \leq A$.

The provisos in (c) and (d) hold even in quantum logic.

These are all generalized versions of parts of Leibniz’ principles of the identity of indiscernibles and its converse. Consider these syntactic analogues;

- (a) $a = b$ implies Rab if and only if Raa is logically true
- (b) $a = b$ implies that Fa and Fb imply each other
- (c) $a = b$ is equivalent to the conjunction of all sentences of form
 $Fa \equiv Fb$
- (d) $a = b$ and Fa together imply Fb
 $a = b$ and Fb together imply Fa .

The only thing we need for such desirable theorems (which contain as good as the whole theory of identity) is to find a transformation f for which we can say: “(Cicero = Tully)” expresses the identity proposition of f.

But there is an obvious candidate: the *variation* (a-b) in which a, b are the abstractors which are labelled by the names “Cicero” and “Tully”. Let us write simply “Iab” for “I((a-b))”. Then the above theorem tells us certainly that most of what we want to say about Iab definitely holds. There is more that is needed or desired, and here is my wish-list:

- (a) Iab is in effect a transitive, reflexive, symmetric relation:
 $Iaa = K$
 $Iab = Iba$
 $Iab \wedge Ibc \leq Iac$
- (b) if a,b are not the same, $Iac = (c-b)Iab$ and $Icb = (c-a)Iab$
- (c) $Iab = K$ only if a,b are the *same* abstractor
- (d) Even if $Iab \neq K$, it still need not imply everything.

All this *except* (d) follows from our definition and the properties of variations (subject to the *modus ponens* proviso $A \wedge (A \rightarrow B) \leq B$ for the last part of (a)). However we can also show that (d) holds in non-trivial models, and that there are such models.⁴

16. Let’s step back now, and see what this means about identity. The proposition expressed by “Cicero = Tully” was identifiable – the meet of all propositions that become true *a priori* if their dependence on Cicero is changed to dependence on Tully. But Cicero, the semantic value of “Cicero”, was not an entity, not the person Cicero, but an abstractor, something active and mobile, a small army of transformations acting on propositions.

This had to be so, since we set out to do *subjective* semantics, in which there can be no use of the person Cicero at all. The abstractor is, if you like, a concept – such a concept as traditionally, but not recently, have been associated with names – but it is not at all the individual concept of a person who is Cicero, nor is it the concept of satisfying descriptions of Cicero. It is instead the action of abstracting *from* Cicero, the action of turning a proposition “about Cicero” into a large family of propositions (of which the original may be one of course) that are not about Cicero (at least in the same way).

17. The project of a Pure Logic has been approached here in the semantic way, that is, by discussing what the most general models for language should be. The building blocks were minds, whose only activity is to suppose, and to conceive of other minds which differ from themselves in what they suppose. From these blocks we can build lattices of proposi-

⁴ QUAAM, (11-10) says, if a,b are distinct then $g^a Iab = Igb$. In this case $Iab \neq K$; it is the minimal proposition Φ only if Igb is (since $g^b(\Phi) = \Phi$ if b is irrelevant to Φ , which it must be if anything is). So the conditions of interest is that g^b should not turn incompatible propositions into K – without which the model would be useless. See also the Appendix of QUAAM for concrete models.

tions. If we then ask what it is like, in detail, for a mind to suppose something, we have to look as the strange case of supposing a general proposition – something we do even in elementary forms of abstract reasoning. This led to the introduction of a new set of important items: the transformations of those lattices of propositions. Using sets of such transformations we can reconstruct quantification, identity, and abstraction. The pleasing result, for whose sake in part the project is undertaken, is that identity propositions are not constrained to have the same status as either tautologies or self contradictions.

APPENDIX (to section 9)

It is quite easy to explain intuitionistic logic in algebraic terms. In each model, the propositions form a lattice, with minimum and maximum K , and a binary operation \rightarrow such that the Great Law of Implication holds:

(*Imp*) A implies $B \rightarrow C$ exactly if $A \wedge B$ implies C

where “implies” denotes the partial ordering of the lattice. The pseudo-complement is defined by: $\neg A = (A \rightarrow \Phi)$. It follows automatically that these lattices are distributive; they are called Heyting lattices.

To check that the propositions in an I-frame form a Heyting lattice, we must show that *Imp* holds. Suppose first that $A \subseteq B \rightarrow C$, i.e. that if $x \in A$ then $x^B \in C$. *A fortiori*, if $x \in A \cap B$ it follows also that $x^B \in C$; but in that case $x = x^B$, so $x \in C$. That establishes *Imp* in one direction. Conversely, suppose that $A \cap B \subseteq C$ and that x is in A . Then clearly $x^B \in A \cap B$ and hence in C , so $x \in B \rightarrow C$.

More interesting is the precondition that $A \rightarrow B$ must be closed if A and B are. This means: any mind subordinate to $A \rightarrow B$ is also a member of it. Suppose therefore that x is subordinate to $A \rightarrow B$. By the special condition on I-frames, this entails that x is also A -subordinate to $A \rightarrow B$. Therefore, for any proposition C , if all $y^A \in C$ for all y in $A \rightarrow B$, then also $x^A \in C$. *A fortiori*, if all $y^A \in B$ for all y in $A \rightarrow B$, it follows that $x^A \in B$. But the antecedent is true by definition; and the consequent is the conclusion that indeed, x is in $A \rightarrow B$.

Turning now to the completeness question, we note that the lattice of propositions in an I-frame is always a complete Heyting lattice. However, every Heyting lattice can be embedded in a complete one, so this feature does not obstruct completeness.

Suppose then that L is a complete Heyting lattice, and define $F(L)$ to be the frame whose elements are the elements of L and whose subordination relation is defined by: x is subordinate to U exactly if $x \leq \vee U$, the complete join of elements of U . In that case $[U] = [\vee U]$, i.e. the principal ideal generated by the element $\vee U$. By another standard theorem (see e.g. R. Balbes and P. Dwinger, *Distributive Lattices*, Ch. IX, section

4, thm. 2) L is isomorphic to the lattice of its own principal ideals. Thus we have L isomorphic to the lattice of propositions of frame $F(L)$, for a proposition is a closed set. Note then that if P is a proposition, then $x \in P$ iff $x \leq \bigvee P$.

In the frame $F(L)$ define $x^A = x \wedge (\bigvee A)$. Then $[x^A] = [x] \cap A$ as required, if A is a proposition. Next suppose that x is subordinate to U , i.e. $x \leq \bigvee U$, and let us try to prove that it is then also A -subordinate to U . So let C be a proposition such that for all y in U , y^A is in C . Because A is a principal ideal, let $A = [a]$, and let $C = [c]$. Thus we have for all y in U , $y \wedge a \leq c$. It follows that $(\bigvee U) \wedge a \leq c$ and hence that $x \wedge a \leq c$, i.e. that x^A is in C . Thus we conclude that $F(L)$ is indeed an I-frame.

Turning to orthologic, our work is made easy by an article of Rob Goldblatt. This is a logic which has a negation; besides the principles reflecting lattice laws it has only

$$\begin{aligned} P \wedge \sim P &\text{ implies } Q \\ P &\text{ implies } \sim \sim P \text{ and conversely} \\ \text{if } P &\text{ implies } Q \text{ then } \sim Q \text{ implies } \sim P \end{aligned}$$

as fundamental principles. Goldblatt provides a semantic analysis in terms of what he calls orthoframes (studied earlier by Foulis and Randall under the name *orthogonality spaces*). The proof of soundness and completeness for orthologic with respect to O-frames follows of course the same pattern as the above proof for intuitionistic logic, except that subordination is defined in terms of the orthogonality relation: x is subordinate to W , by definition, exactly if for all y , $y \perp W$ only if $y \perp x$. Thereafter we can follow Goldblatt's paper so closely that we need not give details here.

As to the classical case, a C-frame is an I-frame in which $x \perp A$ can be defined as: x is subordinate to $A \rightarrow \emptyset$. The latter proposition is the intuitionistic negation of A . If we now require this frame to be also an O-frame, then the law of double negation is imported which reduces intuitionistic logic to classical logic. Completeness is not sacrificed in this way of providing a semantic analysis for classical logic. The reason lies in the fact that every Boolean algebra is isomorphic to the set of elements that equal their own double pseudo-complement ("regular elements") in a Heyting lattice.

Bibliography

- BALBES, R., and P. DWINGER
1974 *Distributive Lattices*, Columbia, Missouri, University of Missouri.
- DALLA CHIARA, M.L.
1977 "Quantum Logic and Physical Modalities", *Journal of Philosophical Logic* 6, 391-404.
- ELLIS, B.
1976 "Epistemic Foundations of Logic", *Journal of Philosophical Logic* 5, 187-204.
1979 *Rational Belief Systems*, Oxford, Blackwell.
- FIELD, H.
1977 "Logic, meaning, and conceptual role", *Journal of Philosophy* 74, 374-409.
- GOLDBLATT, R.
1974 "Semantic Analysis of Orthologic", *Journal of Philosophical Logic* 3, 19-35.
- HARPER, W.L.
1976 "Counterfactuals and representations of rational belief", doctoral dissertation; University of Rochester.
- VAN FRAASSEN, B.C.
1973 "Extension, Intension, and Comprehension", in M. Munitz (ed.) *Logic and Ontology*, New York, New York University Press.
1980 "Critical Study of Ellis (1979)", *Canadian Journal of Philosophy* 10, 497-511.
1982 "Quantification as an Act of Mind", *Journal of Philosophical Logic* 11, 343-369.

