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GENTLEMEN'S WAGERS: RELEVANT LOGIC AND PROBABILITY

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Probability theory and logic are intimately related. If the former is developed as for example by Carnap, logical relations enter into probability axioms (such as: $p(A) = 1$ if A is logically true). Use of an alternative logic will then result in an alternative probability theory. On the other hand, current activity in 'probabilistic semantics' follows Popper's lead: lay down autonomous postulates for probability, define implication probabilistically, and deduce the laws of logic. Choice of alternative probability postulates must then lead to alternative logics.

Here I shall explore some ideas that relate probabilities to relevant logic, hoping to shed some light on both. The impetus came from David Lewis's recent criticism of the semantics for relevant logic offered by, for example, Richard Routley.

Lewis asserts that no sentence can be both true and false. Since $\sim A$ is true (false) just in case A is false (true) and a conjunction is true just in case both its conjuncts are, it follows from this point that $(A \,\&\, \sim A)$ cannot be true. Hence, Lewis points out, no one can object to the inference of B from $(A \,\&\, \sim A)$ that it fails to preserve truth.

I am not inclined to take issue with this point. But Lewis follows up this criticism with further arguments which appear to leave little room for relevant logic except as a logic for equivocaters. The combination of probability with relevant logic may help to provide some defense against this critique, for in the variety of probabilistic implication I construct, $(A \,\&\, \sim A)$ will not only receive probability *zero*, but will be incapable of receiving higher probability, unlike any statement that could ever be true.

I shall introduce the probabilities by means of a sort of conditional bet ('the gentlemen's wager'). The semantics will proceed by associating with each statement a wager, and defining implications in terms of expectations of gain and loss. The exposition will be informal to begin, and the logic defended in this way will be the first degree entailment fragment of R-

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mingle (RMO for short) which was one of the two logics singled out for criticism by Lewis.

I

Bets with a bookie are purchases: I buy a two-dollar bet on Spectacular Bid to win in the first race, at odds 2:1, and receive four dollars if he wins. I have lost the purchase price if he loses. Perhaps I get my money back if the race is called off; that depends on the conditions of sale.

A *gentleman's wager* is somewhat different: no money changes hands until and unless the race is run and won or lost. Similarly for other events we can bet on:

(1) If the Gentlemen win when next they play at Lords', you shall give me a magnum of champagne; if they lose I shall give you a vial of perfume.

Note that two possible events are specified; they are mutually incompatible; one settles the wager in my favour, the other in yours. Remark also that the wager may never be settled; the Gentlemen may never play at Lords' again.

In the above example, let us call the speaker the player who takes the Positive, and the other the Negative. The proposition on which they bet is

(2) The Gentlemen will win the next time they play at Lords'.

If that proposition is A, let us call the two settling events a and $\bar{a}$. There are three probabilities that play some role in the decision whether to accept the wager:

(3) The probability that the wager is settled in favour of the Positive equals $p(a)$; in favour of the Negative, $p(\bar{a})$; that it will be settled at all has probability $p(a) + p(\bar{a})$.

Two of these suffice to determine the third of course.

Let us now compare two bets, on statements A and B. I shall say that A *implies* B if, the stakes being the same, the Positive expectation of B is no less, and its Negative expectation no more, than that of A. This amounts to:

(4) A *implies* B exactly if, for any probability function p,
$$p(a) \leqslant p(b)$$
and $$p(\bar{b}) \leqslant p(\bar{a}).$$

This means that, the stakes being the same, I do not prefer a bet on A to a bet on B, if I am offered the Positive side, and conversely for the Negative. An example is this:

(5) That the Gentlemen will win by five runs the next time they play at Lords' *implies* that the Gentlemen will win the next time they play at Lords'.

The Positive player is surer to get his champagne if they bet on the second proposition, the Negative surer of his perfume if they bet on the first.

It is quite easy to see how logical connectives operate in these bets. The bet on $\sim A$ is just like the bet on A except that the players change sides. That is, if $\bar{a}$ occurs, this new bet is settled for the Positive, and if a occurs, for the Negative. If we bet on $(A \,\&\, B)$, the Positive player wins if and only if both the bet on A and the bet on B are settled for the Positive. The Negative player, on the other hand, wins as soon as *either* of those two is settled for the Negative.

The joint occurrence of a and b may be described as two events occurring, or as the occurrence of a complex event ab of which both are parts. Because I wish to assign probabilities, I need to deal with event-types (to say that an accident did not take place means presumably that the event-type *accident* was not instantiated) because to assign a probability is to assign a number to some thing, and that something needs to be real. I make this explicit here, because I want to say that the bet on $(A \,\&\, \sim A)$ is settled for the Positive exactly if a and $\bar{a}$ occur jointly, and that this joint occurrence has probability *zero*. To make that coherent, I assign this *zero* to the event-type (which has no actually occurring instances), $a\bar{a}$.

To generalize then, if $a_1, \ldots, a_n$ are event (-type)s, so is the complex $a_1 \ldots a_n$, which occurs if and only if $a_1, \ldots, a_n$ occur. The order and collation of the parts do not matter: $(ca)b$, $c(ab)c$, and abc are the same event-type. When a is part of (complex event) e, I shall also say that e *forces* a. Thus forcing is a relation among event-types; $a\bar{a}$ forces itself and a, and also forces $\bar{a}$, but need not force anything else. The other important relation is *incompatibility*: for a and $\bar{a}$ we must choose incompatible events; these cannot both occur.

(6) If A is a statement $\langle A \rangle$ is the wager on this statement (for stakes unspecified) which may be settled for the Positive or for the Negative by the rules below.

(7) If A is atomic there are associated mutually incompatible events a and $\bar{a}$; $\langle A \rangle$ is settled for the Positive by any event which forces a and for the Negative by any event that forces $\bar{a}$.

(8) $\langle \sim A \rangle$ is settled for the Positive by any event that settles $\langle A \rangle$ for the Negative, and $\langle \sim A \rangle$ is settled for the Negative by any event that settles $\langle A \rangle$ for the Positive.

(9) An event settles $\langle A \,\&\, B \rangle$ for the Positive exactly if it settles both $\langle A \rangle$ and $\langle B \rangle$ for the Positive; for the Negative if it settles either $\langle A \rangle$ or $\langle B \rangle$ for the Negative.

(10) An event settles $\langle A \vee B \rangle$ for the Positive exactly if it settles either $\langle A \rangle$ or $\langle B \rangle$ for the Positive; and for the Negative if it settles both $\langle A \rangle$ or $\langle B \rangle$ for the Negative.

These conditions were written at length to emphasize the intuitive content. If we use the notations $\langle A |$ and $| A \rangle$ for the sets of event(-type)s that settle $\langle A \rangle$ for the Positive and Negative respectively, we see at once that these are combined by set intersection and set union to represent conjunction and disjunction. It is an easy lemma to prove that all the sets $\langle A |$ and $| A \rangle$ are *closed sets* in the sense:

(11) A set X of events is *closed* exactly if it contains all the events that force any of its members.

In view of their use, I shall call the closed sets *propositions*. (The simplicity of the closure operation here guarantees that unions of closed sets are also closed.) A statement is associated with a bet, and thereby with a pair of propositions.

How shall we assign probabilities for winning and losing such a bet? The Positive wins the bet on A exactly if a occurs, which is exactly if some event occurs that forces a. Probability must be probability of occurrence.

I. The probability that some event which forces a occurs, equals the probability of a itself, $p(a)$.

II. If a and b are incompatible, and e forces both, then e cannot occur, and hence $p(e) = 0$.

This is a beginning, and may or may not be enough for the treatment of atomic statements. We are relying on previous knowledge about how to assign probabilities to events, and (I) tells us how to extend them to certain propositions. Classical probability theory and even certain generalizations (notably Birkhoff's valuations on lattices) insist that extension to conjunctions and disjunctions be via an additivity principle.

III. $p(X \cup Y) + p(X \cap Y) = p(X) + p(Y)$ for any propositions X and Y.

Given our very simple 'settlement conditions' (7)–(10), this suffices to extend probabilities to $\langle A |$ and $| A \rangle$ for each statement A in propositional logic.

The reader is now invited to a wager on a contradiction. No doubt he prefers the Negative. If he is an unmitigated classical logician, he may not think there is much more to be said. But I shall offer him a choice of wagers, on different contradictions. Let the statement A be successively

A_1. Tomorrow's sunrise will be before 8 a.m.;

A_2. The first human to land on a planet outside our galaxy will find life there;

A_3. The end of the world will occur on a Wednesday.

By the principle of the gentlemen's wager, no money or goods will change hands until or unless one of the events occurs which settles the wager. To settle the bet on $(A \,\&\, \sim A)$, we must wait till either $a\bar{a}$, or a, or $\bar{a}$ occurs. The first of these will not happen at all, of course, but the others may not either. In the first example, $\langle A_1 \,\&\, \sim A_1 \rangle$ will be settled at tomorrow's sunrise, in the second, at the astronaut's landing, and in the third, at the end of the world. It is not merely a matter of waiting a long time; we are also much less sure that there will be such astronautical exploits, or that the world will ever end. So taking the Negative on $\langle A_1 \,\&\, \sim A_1 \rangle$ is much to be preferred to taking the Negative on $\langle A_2 \,\&\, \sim A_2 \rangle$.

Dually, I would prefer to be offered a chance to take the Positive side on $\langle A_1 \vee \sim A_1 \rangle$ rather than on $\langle A_2 \vee \sim A_2 \rangle$. It is easy to see why, since payoff for the Positive side on an excluded middle is clearly the same as for the Negative side on the corresponding explicit contradiction. If there is only a small probability that $\langle A \rangle$ will be settled either way, all expectations on bets whose settlement depends on the settlement of $\langle A \rangle$ will also diminish.

But of course the Positive expectation on $\langle B \vee \sim B \rangle$ must be at least as high as the Positive expectation on $\langle A \,\&\, \sim A \rangle$, and conversely for the Negative. Hence although contradictions do not generally imply each other, they do imply all excluded middles:

$(A \,\&\, {\sim}A)$ Implies $(B \vee {\sim}B)$

which is the characteristic feature of R-mingle. The proof that RMO does indeed capture all and only the valid implications we have described needs and deserves a more formal treatment.

II

Before making the samantics more precise, let us take a look at the two logical systems to which the discussion pertains. The first was called, by its creators Anderson and Belnap, *the logic of tautological entailment*. Using the terminology according to which a formula without implication connectors is of zero-degree, and an entailment between such, of first degree, that logic is also the first degree fragment of their most famous logical system E, and so I shall also call it E0.

This is a very simple logic (see Anderson and Belnap, Section 15.2). The first rule says that entailment is transitive. Next come the usual 'lattice' principles for conjunction and disjunction:

$A \,\&\, B$ entails A; $A \,\&\, B$ entails B.
If A entails B and also entails C then A entails $B \,\&\, C$.
A entails $A \vee B$; B entails $A \vee B$.
If A entails C and B entails C then $A \vee B$ entails C.

Next comes the Distribution Principle: $A \,\&(B \vee C)$ entails $(A \,\&\, B) \vee (A \,\&\, C)$. And finally we have the principles for negation:

A and ${\sim\sim}A$ entail each other.
If A entails B then ${\sim}B$ entails ${\sim}A$.

At this point it may well be asked how much of classical logic is missing. In one sense, not much. Specifically, this logic has exactly the same procedure for turning any formula into a logically equivalent one that is in normal form, whether disjunctive or conjunctive. This is done both in E0 and in classical logic by application of the principles of equivalence: Commutation (of &, ∨); Associativity (of &, ∨); Distribution (in both directions and both dual forms); Double Negation; and De Morgan's Laws.

Yet in this logic, no fallacies of relevance can be committed. $(B \,\&\, {\sim}B)$ does not entail everything; it does not entail A, nor even $(A \vee {\sim}A)$ for arbitrary A. It is at first sight startling, to the classical eye, to find that $A \,\&\, ({\sim}A \vee B)$ does not entail B. But on the one hand, C. I. Lewis already gave one argument that shows that if we accepted this as an entailment, in addition to the above principles, then $(B \,\&\, {\sim}B)$ would entail A after all. And on the other, while *modus ponens* should hold for any respectable implication connector, who would call material implication respectable?

The second logical system is the first degree fragment RMO of the logic *R-mingle*. This consists of the logic E0 plus a single extra rule:

$(B \,\&\, {\sim}B)$ entails $(A \vee {\sim}A)$

While that is a clear fallacy of relevance, the logic is still of the relevance species, for it is still not the case that $(B \,\&\, {\sim}B)$ entails every sentence, nor that every sentence entails $(A \vee {\sim}A)$. It is not even the case that all contradictions entail each other, nor that tautologies all entail each other. So, many of the distinctions of relevance that transcend classical logical equivalence, are respected in this logic.

III

What happens to probability on a relevant logic? The logics I single out for study are the first degree entailment fragments of E (logic of tautological entailment, or E0) and of R-mingle (RMO). It will turn out that the fine structure of the former can not be reflected entirely in the probability functions. The case is different for RMO; and indeed, RMO appears as a sort of probabilistic reduction of E0.

The natural family of structures for the semantics of E0 is that of De Morgan lattices. In the completeness proof we find, as usual, that considerably smaller classes already reflect the entire structure of the logic – though perhaps not its possibilities for extension to E and R – and I shall restrict my modest efforts here to one such class (essentially, the one studied in my [1969], the exposition modified using my [1973] and Dunn's [1976]).

A (*finite, free*) *event structure* is a triple $F = \langle FA, FE, - \rangle$ in which FA is a finite class of sets, FE is the power set of FA, and – is an operation on FA with the simple property that a and $\bar{a}$ are disjoint sets.

We call FE the class of *events*. The idea is that FA is itself a class of events $a_1, a_2, \ldots$ and that $\{a_1, \ldots, a_k\}$ represents the complex event which is the joint occurrence or combination of a_1 and ... a_k. Recall also that I mean 'event' here in the sense of 'event type'. We think of a and $\bar{a}$ as incompatible events (this being represented by their disjointness) but the event type $\{a, \bar{a}\}$ is different from the event type $\{b, \bar{b}\}$ even though each represents an impossible joint occurrence.

It is convenient to concentrate on the structure of FE; the role of event a is taken over by that of $\{a\}$ which represents the joint occurrence of a with itself. The operation – will of course be used in the treatment of negation, but that subject I will postpone for a while.

Event e (member of FE) is said to *force* event e' exactly if $e' \subseteq e$. And we write ee' for $e \cup e'$, the operation of conjunctive combination of events. We shall now construct the *propositions* on our event structure.

(1) A set of events E is *closed* exactly if each event that forces some member of E also belongs to E; the *closure* $[E]$ of E is the least closed set that contains it; a *proposition* is a closed set of events.

Two conventions will facilitate writing: I shall use $X, Y, Z, X', \ldots$ to stand for propositions and $E, E', E_1, \ldots$ for arbitrary sets of events and abbreviate $[\{e_1, \ldots, e_n\}]$ to $[e_1, \ldots, e_n]$. It is an easy lemma that intersections and unions of closed sets are closed again (due to the especially simple nature of our closure operation) so the propositions automatically form a disttibutive lattice. More simply, we note:

(2) $[e] \cap [e'] = [ee']$,
$[e] \cup [e'] = [e, e']$,
$[e] \cap [e_1, \ldots, e_n] = [e_1 e, \ldots, e_n e]$.

(3) The *base* X_0 of X is the smallest set of events whose closure equals X; the *rank* rX equals the cardinality of its base.

To justify this definition let us call a set of events *redundant* if it has two members of which one forces the other. All our sets are finite, so we can reduce any set to a non-redundant one which still has the same closure by successively tossing out events which force some other (remaining) member of the set. The result of this is unique (and the uniqueness argument does not depend on finitude): for suppose that E and E' are non-redundant sets with the same closure. If e_1 is in E and not in E' it must then force some event e_2 in E'. The latter cannot be in E on pain of redundancy. But it is in $[E]$ so it

must force some event e_3 in E. However, forcing is transitive, so e_1 forces e_3 and hence E is redundant after all. Therefore we conclude that $E = E'$.

We are now ready to introduce probabilities. Looking at the 'atomic' events in FA we see that they are represented by sets, so we begin by choosing an ordinary probability function defined on those.

(4) A *pre-probability* on F is a probability function whose domain contains FA.

We want probability to be probability of occurrence, and so the probability given to combination event $\{a, b\}$ must be the probability of the joint occurrence of a and b. But the choice of p already rules on that: it is $p(a \cap b)$. Hence we can extend p to FE in only one way:

(5) The extension of p to FE is defined by $p(e) = p(\cap e)$.

While $\cap e$ is not usually in FA, it will of course be in the domain of p, since we are only looking here at finite event structures.

The probability that event e occurs must be the same as the probability that some event which forces e occurs – for e occurs if and only if that happens. Hence the first step in assigning probabilities to propositions is also quite out of our hands.

(6) The extension of p to propositions must be such that $p([e]) = p(e)$.

Next we insist that this extension must *be* a probability function, and that requires at the very least that it satisfies a postulate of additivity. The most orthodox of these is:

(7) $p(X \cup Y) + p(X \cap Y) = p(X) + p(Y)$.

Now probability functions have more properties than just additivity. Looking at Birkhoff's classic treatment of valuations on lattices, we see the additional postulate that p must preserve the order (if $X \leqslant Y$ then $p(X) \leqslant p(Y)$) and that the value of p is between *zero* and *one* inclusive. Order preservation especially is a feature that must be separately postulated when we move out of the comfortable classical environment of Boolean algebras. But whatever may have to be done in more general cases, we have already said as much as we can for, as I shall show, the probabilities of the propositions are at this point uniquely determined. To allay our fears that the extension does not deserve to be called a probability function after all, I shall prove a bit more.

Theorem. The extension of p to propositions is uniquely determined by (5)–(7); $0 \leqslant p(X) \leqslant 1$; and if $X \leqslant Y$ then $p(X) \leqslant p(Y)$.

To prove this we define a mapping of the propositions into the field of sets generated by FA (which must still be part of the domain of p):

$$f(e) = \cap\, e$$
$$f(X) = \cup\, \{f(e)\colon e \text{ in } X\}$$

We prove first the lemma that $p(X) = p(f(X))$. This is clearly so for Λ and for X of rank 1, by (6). Suppose it is the case for all propositions of rank $\leqslant n$ and let Y have base $\{e_1\ \ldots, e_n, e\}$; write $Y = [E] \cup [e]$. Then by (2) and (7), we have

$$p(Y) = p([E]) + p([e]) - p([e_1 e, \ldots, e_n e]).$$

Since each of the propositions on the right hand side are closure of sets of cardinality $\leqslant n$, they have rank $\leqslant n$ and we conclude

$$\begin{aligned} p(Y) &= p(f([E])) + p(f[e]) - p(f([E]) \cap f([e])) \\ &= p(f([E]) \cup f([e])) \\ &= p(f([E] \cup [e])) \\ &= p(f(Y)). \end{aligned}$$

This shows the first, and as corollary the second, part of the theorem. To prove the third it suffices now to realize if $X \leqslant Y$ then $f(X) \leqslant f(Y)$, which follows directly from the definition of f.

IV

We turn now to the treatment of negation and implication. It may be recalled that in Section I, we associated with atomic sentence A the couple of propositions $[\{a\}]$ and $[\{\overline{a}\}]$. There must be some operation that turns the one into the other; and we may wonder whether that is not generally the case for $\langle A\,|$ and $|\,A\rangle$ (which is, $\langle \sim A|$). There is no 'complementation' operation on events in general but we can define

1. e is orthogonal to e' (briefly, $e \perp e'$) exactly if there is a member a of FA such that e forces $\{a\}$ and e' forces $\{\overline{a}\}$ or conversely.

Of course FA is the 'strongest' event which forces all events; it is also orthogonal to all events except Λ. In addition, orthogonality is symmetric. But there are *many* events that are self-orthogonal, so this is not exactly like the geometric relation.

2. $E^{\perp}$ (read as 'E-perp') is the set of all events that are orthogonal to all members of E.

We can now state the theorem that allows us to think of each sentence as expressing a single proposition.

Theorem. If X is a proposition, so is $X^{\perp}$; $\langle \sim A\,| = \langle A\,|^{\perp} = |A\rangle$; the closed sets with $\cap, \cup, {}^{\perp}$ form a De Morgan lattice.

Adding the first and second part of this theorem to my earlier papers immediately yields the third part. Moreover, since any proposition can be $<A|$ and since $<\sim A|$ is also a proposition, proof of the second part proves the first as well.

That $[\{\overline{a}\}] = \{e\colon e \text{ forces } \overline{a}\}$ is the set of all facts that are orthogonal to all those that force $\{a\}$, is clear. Similarly with a and a interchanged. Hence if A is atomic, $<A|^{\perp} = |A>$ and $|A>^{\perp} = <A|$. Let us take this feature for our hypothesis for sentences A up to given complexity. If $B = \sim A$, it clearly also has the feature. To finish the introduction, we need only prove De Morgan's Laws:

3. $(X \cap Y)^{\perp} = X^{\perp} \cup Y^{\perp}$
4. $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$

If e is orthogonal to all members of X (respectively, of Y) it is clearly orthogonal to all members of $X \cap Y$. Secondly, suppose e is orthogonal to all members of $X \cap Y$. Now suppose e' is in X and e is not orthogonal to it. Now let e'' be in Y. Then $e'e''$ is in $X \cap Y$, so e is orthogonal to that. But if e forces $\{b\}$ and $e'e''$ forces $\{\overline{b}\}$ that must then be because e'' forces $\{\overline{b}\}$. We conclude that if e fails to be orthogonal to some event in X, then it is orthogonal to all events in Y. I leave the proof for the dual case to the reader.

How does this sort of negation interact with probability? Well, joint occurrences of orthogonal events *must* receive probability zero, given our method of assignment, and hence so do all self-orthogonal events.

5. If $e \perp e'$ then $p(ee') = 0$.
6. If $X \subseteq Y^{\perp}$ then $p(X \cup Y) = p(X) + p(Y)$.

Kolmogorov's negation axiom is recalled by 6, which follows directly from 5. It is customary to call X and Y *orthogonal* too if $X \subseteq Y^{\perp}$ (a symmetric relationship).

Our gentlemen's wagers induced a variety of implication, which we can now formulate with precision.

7. *X implies Y in p* exactly if $p(X) \subseteq p(Y)$ and $p(Y^{\perp}) \subseteq p(X^{\perp})$.
8. *X probabilistically implies Y* (briefly $X \vdash_p Y$) exactly if X implies Y in all probability functions p.

In a De Morgan lattice, $-Y \subseteq -X$ if $X \subseteq Y$, and we found above that p preserves the order. Hence we conclude that if $X \subseteq Y$ then $X \vdash_p Y$. Since the logical principles of E0 are valid in all De Morgan lattices, in the ordinary sense, it now follows that they are all valid also in the present probabilistic sense. But we have an addition.

Theorem. If $A \vdash B$ in RMO then $\langle A| \vdash_p \langle B|$.

This follows from the above, plus

9. $X \cap X^{\perp} \vdash_p Y \cup Y^{\perp}$.

That is proved by 6 which implies that $p(X_1 \cap X_1^{\perp}) = 0$, and the fact that De Morgan's laws hold.

Having now found therefore that RMO is sound for probabilistic implication, we turn to completeness. In E0, every formula can be transformed into an equivalent disjunctive normal form and also into a conjunctive one. Since E0 is part of RMO we can begin with that procedure. So we need to check for validity only cases that take the following form:

10. $A \vdash B$ where
$A = A_1 \vee \ldots \vee A_m$,
$B = B_1 \,\&\, \ldots \,\&\, B_n$,
$A_i = A_i^1 \,\&\, \ldots \,\&\, A_i^q$,
$B_j = B_j^1 \vee \ldots \vee B_j^r$.
Each sentence A_i^s and B_j^t is either an atomic sentence or the negation thereof.

We need to prove that if $A \vdash B$ does not hold in RMO, neither does $\langle A| \vdash_p \langle B|$. I shall here write $p(A)$ for $p(\langle A|)$, and so forth, for brevity.

The derivability of B from A in RMO is easy to check when they are in

the above normal form. It is required exactly that *each* couple $\langle A_i, B_j \rangle$ be *good* in the following sense:

either

$$A_i^s = B_j^t \text{ for some } s, t$$

or

$$A_i^s = {\sim}A_i^u \text{ and } B_j^t = {\sim}B_j^w \text{ for some } s, u, t, w.$$

Let us suppose, to be definite, that the couple $\langle A_1, B_1 \rangle$ is not good in this sense. We shall first of all show that $A_1 \vdash_p B_1$ does not hold.

To begin, we attempt to choose a single atomic event a and set $\langle A_1^s| = [\{a\}]$ for all s. If that succeeds, we choose two events b and $\bar{b}$, disjoint from each other and *also* from a, and set $\langle B_1^t|$ equal to $[\{b\}]$ if B_1^t is an atomic sentence and equal to $[\{\bar{b}\}]$ otherwise. Then we choose probability function p so that $p(a) = 1$ and $p(b) = p(\bar{b}) = 0$. This implies that $p(A_1) = 1$ and $p(B_1) = 0$ so A_1 does not probabilistically imply B_1. Should the attempt fail, it must be because A_1^s is the same as ${\sim}A_1^u$ for some s and u. In that case, because the couple is not good, no B_1^t is the same as ${\sim}B_1^w$ for any t and w. We then recall that in RMO, $A_1 \vdash B_1$ only if ${\sim}B_1 \vdash {\sim}A_1$ and set out to refute the latter. Placed in normal form we have

$$\begin{gathered}{\sim}B_1 \vdash {\sim}A_1 \text{ if and only if}\\ {*}B_1^1 \,\&\, \ldots \,\&\, {*}B_1^r \vdash {*}A_1^1 \vee \ldots \vee {*}A_1^q,\end{gathered}$$

where ${*}C$ is ${\sim}C$ if C is unnegated, and is D if C is ${\sim}D$. Now we try the same strategy as above, and this time it cannot fail. But $\langle {\sim}C| = \langle C|^{\perp}$, and so we see that A_1 does not probabilistically imply B_1 in this case either.

We have now found that since $A_1 \vdash B_1$ does not hold, there is a probability function (on some well-chosen event-structure) such that either $p(A_1) = 1$ and $p(B_1) = 0$, or else $p({\sim}B_1) = 1$ and $p({\sim}A_1) = 0$. In the former case, we have also $p(A) > p(B)$. In the latter case we note that $A \vdash B$ holds only if ${\sim}B \vdash {\sim}A$ does where that can be put in dual normal form

$${*}B_1 \vee \ldots \vee {*}B_n \vdash {*}A_1 \,\&\, \ldots \,\&\, {*}A_m,$$

and we shall have $p({*}B_1) > p({*}A_1)$ and hence $p({\sim}B) > p({\sim}A)$ and so again A does not probabilistically imply B.

V

Logic catalogues valid arguments; in doing so it characterizes one aspect of the structure of a language. That is usually a fairly superficial aspect, which may be shared by many languages of quite different structure. In our case that is quite clear from the completeness proof: the restriction of our probability functions to those having values *zero* and *one* only (truthvalue functions) would not upset that proof at all. To exhibit RMO as a sort of reduced E0 it would suffice, this suggests, to concentrate on maps that assign each proposition in a De Morgan lattice the value *zero* or *one* in a way that preserves the implicational order and does not assign *one* to any proposition and its negate, and defining *t*-implication analogously to the probabilistic implication. David Lewis's criticisms could be answered in connection with this manoeuvre too.

But the corollaries for logic aside, we have found out something about the relation between probability and logic. There is an important difference between the conditional bets that are used in gentlemen's wagers and the ordinary bets. Each gives rise to a probability calculus, but the two are different because the latter calculus leads to a classical logic and the former to a relevant logic. Because I studied the matter by looking at how relevant probability functions can be constructed, there are still questions to be answered about axiomatics. For example, both Intuitionistic logic and RMO are characterized by distributive lattices which have a sort of non-standard complementation operation. In both cases, the probabilities defined on the lattice then have the properties

$$p(A \mathbin{\&} {\sim}A) = 0; \quad p(A \vee {\sim}A) \leqslant 1.$$

So far the similarities; what are the differences? They have to do with double negation and De Morgan's laws of course. In the case of Intuitionistic logic there is an autonomous probability theory (in the guise of probabilistic semantics for that logic along the lines which Popper proposed for classical logic) from which the Intuitionistic logical principles can be derived (see my [1981]). Is there an autonomous axiomatic probability theory that can play this role for RMO? Can it be extended to the implication connector of RM? Are there interesting stochastic models among our probabilified event structures?

Among probabilified De Morgan lattices generally? These are all questions that seem to me to deserve further study.

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