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For all problems that involve coding, please include your code.

**Problem 1: Accounting for nonlinearity and modeling error in stability analysis**

It is common in control theory to approximate an unknown dynamical system with a linear model, but also to account for nonlinear effects by adding a bounded unknown nonlinear term. More precisely, the dynamics is modelled as

$$x_{k+1} = Ax_k + g(x_k), \quad (1)$$

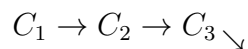
where  $A \in \mathbb{R}^{n \times n}$  is a fixed and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is an unknown continuous function satisfying

$$\|g(x)\| \leq \gamma \|x\| \quad \forall x \in \mathbb{R}^n \quad (2)$$

for some fixed scalar  $\gamma > 0$ . An important problem in control is to check whether  $x = 0$  is a globally asymptotically stable equilibrium point of the dynamics in (1) for any choice of the function  $g$  verifying (2). In order to check this property, one can search for a (homogeneous and coercive) quadratic Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying

$$V(Ax + g(x)) < V(x) \quad \forall x \neq 0, \text{ and for any function } g \text{ verifying (2)}. \quad (3)$$

1. Formulate the search for such a Lyapunov function as an SDP feasibility problem.
2. A series of chemical reactions



between three chemical compounds  $C_1, C_2$ , and  $C_3$  can be modeled by a dynamical system of the type in (1), where  $x_k$  is a  $3 \times 1$  vector whose  $i^{\text{th}}$  component  $x_{k,i}$  represents the concentration of chemical compound  $i$  at time  $k$ . Here, the matrix  $A$  is given by

$$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

representing that at each time step, half of  $C_1$  converts to  $C_2$ , half of  $C_2$  converts to  $C_3$ , and half of  $C_3$  vanishes. The function  $g : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is unknown and represents the hard-to-model nonlinear interactions between the chemical compounds.

What is the largest value of  $\gamma$  (to two digits after the decimal point) such that if

$$\|g(x)\| \leq \gamma \|x\| \quad \forall x \in \mathbb{R}^3,$$

then all chemical concentrations go to zero irrespective of their initial concentrations?

*Hint: To find lower (resp. upper) bounds on this critical value of  $\gamma$ , leverage part 1 (resp. focus on functions  $g$  that are linear).*

### Problem 2: Getting past exponentially many spurious local minima

A polynomial  $p : \mathbb{R}^n \mapsto \mathbb{R}$  is *separable* if it can be written as  $p(x) = \sum_{i=1}^n q_i(x_i)$ , where each  $q_i$  is a univariate polynomial.

- (a) Show that a separable polynomial is nonnegative if and only if it is a sum of squares. (You can use the fact that a univariate nonnegative polynomial is a sum of squares without proof.)
- (b) Present an explicit family of degree-4 polynomials  $p_n : \mathbb{R}^n \mapsto \mathbb{R}$  such that
  - (i) the number of nonglobal local minima of  $p_n$  grows exponentially with  $n$ ,
  - (ii) for all  $n$ , we have

$$\left[ \min_{x \in \mathbb{R}^n} p_n(x) \right] = \left[ \begin{array}{c} \max_{\gamma \in \mathbb{R}} \quad \gamma \\ \text{s.t.} \quad p_n(x) - \gamma \text{ is a sum of squares} \end{array} \right].$$

For context, this means that semidefinite programming can find the exact optimal value of nonconvex problems with exponentially many spurious local minima.

### Problem 3: Equivalence of decision and search for some problems in NP

1. Suppose you had a blackbox that given a 3SAT instance would tell you whether it is satisfiable or not. How can you make polynomially many calls to this blackbox to find a satisfying assignment to any satisfiable instance of 3SAT?
2. Suppose you had a blackbox that given a graph  $G$  and an integer  $k$  would tell you whether  $G$  has a stable set of size larger or equal to  $k$ . How can you make polynomially many calls to this blackbox to find a maximum stable set of a given graph?

**Problem 4: Complexity of rank-constrained SDPs**

Consider a family of decision problems indexed by a positive integer  $k$ :

**RANK- $k$ -SDP**

**Input:** Symmetric  $n \times n$  matrices  $A_1, \dots, A_m$  with entries in  $\mathbb{Q}$ , scalars  $b_1, \dots, b_m \in \mathbb{Q}$ .

**Question:** Is there a real symmetric matrix  $X$  that satisfies the constraints

$$\text{Tr}(A_i X) = b_i, i = 1, \dots, m, X \succeq 0, \text{rank}(X) = k?$$

Show that RANK- $k$ -SDP is NP-hard for any integer  $k \geq 1$ .

(Hint: First show NP-hardness for  $k = 1$ , then see how you can modify your construction so that it would work for any other  $k$ .)

**Problem 5: Concave box QP**

Show that the following decision problem is NP-complete.

**CONCAVE-BOX-QP:** Given a symmetric matrix  $Q \in \mathbb{Q}^{n \times n}$ , with  $Q \preceq 0$ , vectors  $c, l, u \in \mathbb{Q}^n$ , and a scalar  $k \in \mathbb{Q}$ , decide whether the optimal value of the following optimization problem is less than or equal to  $k$ :

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & -l_i \leq x_i \leq u_i \quad i = 1, \dots, n. \end{aligned}$$