ORF 523	Problem set 6	Spring 2025, Princeton University
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For all problems that involve coding, please include your code.

Problem 1: Accounting for nonlinearity and modeling error in stability analysis
It is common in control theory to approximate an unknown dynamical system with a linear model, but also to account for nonlinear effects by adding a bounded unknown nonlinear term. More precisely, the dynamics is modelled as

$$x_{k+1} = Ax_k + g(x_k), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a fixed and $g : \mathbb{R}^n \to \mathbb{R}^n$ is an unknown continuous function satisfying

$$\|g(x)\| \leq \gamma\|x\| \ \forall x \in \mathbb{R}^n \tag{2}$$

for some fixed scalar $\gamma > 0$. An important problem in control is to check whether $x = 0$ is a globally asymptotically stable equilibrium point of the dynamics in (1) for any choice of the function g verifying (2). In order to check this property, one can search for a (homogeneous and coercive) quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$V(Ax + g(x)) < V(x) \quad \forall x \neq 0, \text{ and for any function } g \text{ verifying (2).} \tag{3}$$

1. Formulate the search for such a Lyapunov function as an SDP feasibility problem.

2. A series of chemical reactions

$$C_1 \to C_2 \to C_3 \searrow$$

between three chemical compounds C_1, C_2, and C_3 can be modeled by a dynamical system of the type in (1), where x_k is a 3×1 vector whose i^{th} component $x_{k,i}$ represents the concentration of chemical compound i at time k. Here, the matrix A is given by

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

representing that at each time step, half of C_1 converts to C_2, half of C_2 converts to C_3, and half of C_3 vanishes. The function $g : \mathbb{R}^3 \to \mathbb{R}^3$ is unknown and represents the hard-to-model nonlinear interactions between the chemical compounds.

What is the largest value of γ (to two digits after the decimal point) such that if

$$||g(x)|| \leq \gamma \|x\| \; \forall x \in \mathbb{R}^3,$$

then all chemical concentrations go to zero irrespective of their initial concentrations?

Hint: To find lower (resp. upper) bounds on this critical value of γ, leverage part 1 (resp. focus on functions g that are linear).

Problem 2: Getting past exponentially many spurious local minima

A polynomial $p : \mathbb{R}^n \mapsto \mathbb{R}$ is *separable* if it can be written as $p(x) = \sum_{i=1}^n q_i(x_i)$, where each q_i is a univariate polynomial.

(a) Show that a separable polynomial is nonnegative if and only if it is a sum of squares. (You can use the fact that a univariate nonnegative polynomial is a sum of squares without proof.)

(b) Present an explicit family of degree-4 polynomials $p_n : \mathbb{R}^n \mapsto \mathbb{R}$ such that

(i) the number of nonglobal local minima of p_n grows exponentially with n,

(ii) for all n, we have

$$\left[\min_{x\in\mathbb{R}^n} p_n(x)\right] = \begin{bmatrix} \max\limits_{\gamma\in\mathbb{R}} & & \gamma \\ \text{s.t.} & p_n(x) - \gamma & \text{is a sum of squares} \end{bmatrix}.$$

For context, this means that semidefinite programming can find the exact optimal value of nonconvex problems with exponentially many spurious local minima.

Problem 3: Equivalence of decision and search for some problems in NP

1. Suppose you had a blackbox that given a 3SAT instance would tell you whether it is satisfiable or not. How can you make polynomially many calls to this blackbox to find a satisfying assignment to any satisfiable instance of 3SAT?

2. Suppose you had a blackbox that given a graph G and an integer k would tell you whether G has a stable set of size larger or equal to k. How can you make polynomially many calls to this blackbox to find a maximum stable set of a given graph?

Problem 4: Complexity of rank-constrained SDPs

Consider a family of decision problems indexed by a positive integer k:

RANK-k-SDP
Input: Symmetric $n \times n$ matrices $A_1, \ldots, A_m$ with entries in $\mathbb{Q}$, scalars $b_1, \ldots, b_m \in \mathbb{Q}$.
Question: Is there a real symmetric matrix X that satisfies the constraints

$$\mathrm{Tr}(A_i X) = b_i, i = 1, \ldots, m, \; X \succeq 0, \; \mathrm{rank}(X) = k?$$

Show that RANK-k-SDP is NP-hard for any integer $k \geq 1$.

(Hint: First show NP-hardness for $k = 1$, then see how you can modify your construction so that it would work for any other k.)

Problem 5: Concave box QP

Show that the following decision problem is NP-complete.

CONCAVE-BOX-QP: Given a symmetric matrix $Q \in \mathbb{Q}^{n \times n}$, with $Q \preceq 0$, vectors $c, l, u \in \mathbb{Q}^n$, and a scalar $k \in \mathbb{Q}$, decide whether the optimal value of the following optimization problem is less than or equal to k:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & -l_i \leq x_i \leq u_i \quad i = 1, \ldots, n. \end{aligned}$$