Optimal Expectations*

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Abstract

This paper introduces a tractable, structural model of subjective beliefs. Forward-looking agents care about expected future utility flows, and hence have higher current felicity if they believe that better outcomes are more likely. On the other hand, expectations that are biased towards optimism worsen decision making, leading to poorer realized outcomes on average. Optimal expectations balance these forces by maximizing the lifetime well-being of an agent. We apply our framework of optimal expectations to three different economic settings. In a portfolio choice problem, agents overestimate the return on their investment and may invest in an asset with negative expected excess return if sufficiently positively skewed. In general equilibrium, agents' prior beliefs are endogenously heterogeneous, leading to gambling. Finally, in a consumption-saving problem with stochastic income, agents are both overconfident and overoptimistic, and consume more than implied by rational beliefs early in life.

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1 Introduction

Modern psychology views human behavior as a complex interaction of cognitive and emotional responses to external stimuli that sometimes gives dysfunctional outcomes. Modern economics takes a relatively simple view of human behavior as governed by unlimited cognitive ability applied to a small number of concrete goals and unencumbered by emotion. The central models of economics allow coherent analysis of behavior and of economic policy, but eliminate "dysfunctional" outcomes, and in particular the possibility that households might persistently err in attaining their goals. One area in which there is substantial evidence that households do consistently err is in the assessment of probabilities. In particular, agents often overestimate the probability of good outcomes, such as their success (Alpert and Raiffa (1982), Weinstein (1980), and Buehler, Griffin, and Ross (1994)).

We provide a structural model of subjective beliefs in which agents hold incorrect but optimal beliefs. These optimal beliefs differ from objective beliefs in ways that match many of the claims in the psychology literature about "irrational" behavior. Further, in the canonical economic models that we study, these beliefs lead to economic behaviors that match observed outcomes that have puzzled the economics literature based on rational behavior and common priors. Our approach has three main elements.

First, at any instant, people care about current utility flow and expected future utility flows. We also allow, although it is not central to our main points, that past utility flows also influence current felicity. While it is standard that agents that care about expected future utility plan for the future, forward-looking agents have higher current felicity if they are optimistic about the future. Phrases like "anticipation exceeds realization"[1] are consistent with this idea. Agents that care about expected future utility flows are happier with distorted beliefs about the payoff of their investments and/or the stochastic process for future labor income.

The second crucial element of our model is that such optimism affects actual decisions. Distorted beliefs distort actions. In this sense, agents are not schizophrenic. For example, an agent cannot derive utility by optimistically believing that she will be rich tomorrow, while also

[1] The common German phrase "Vorfreude ist die schönste Freude" translates roughly to "anticipation exceeds realization."

basing today's consumption-savings decision on rational beliefs about future income.

How are these forces balanced? We assume that subjective beliefs maximize the agent's expected total well-being, the expected discounted sum of felicity across periods. This third key element leads to a balance between the first two – the benefits of optimism and the costs of basing actions on distorted expectations.

There are several reasons why beliefs might maximize overall well-being. First, in the long tradition of economics, we want to explore what is optimal. Second, if nature (or parents) care about the happiness of agents, it would choose to endow them with optimal expectations. This is consistent with children being raised believing they can "do or be anything," "they are special," etc. Finally, the scientific test of the theory is its performance, not its assumptions. So far, our approach helps to explain heterogenous beliefs, gambling, overconfidence, procrastination, and intertemporal preference reversal, all within one coherent and tractable model.

We demonstrate how these utility-serving biases distort beliefs and behavior. In general, beliefs are less rational when biases have little cost in realized outcomes and when biases have large benefits in terms of expected future happiness. Beliefs tend towards optimism – states with greater utility flows are perceived as more likely. More specifically, we illustrate our theory of optimal expectations using three examples. First, in a portfolio choice problem, agents overestimate the return of their investment and underdiversify. Second, in general equilibrium, agents' prior beliefs are endogenously heterogeneous and agents gamble against each other. The price of the risky asset may differ from that in an economy populated by agents with rational beliefs. Third, in a consumption-saving problem with quadratic utility and stochastic income, agents are overconfident and overoptimistic; agents consume more than implied by rational beliefs early in life. In addition, Brunnermeier and Parker (2003) show in a different economic setting that agents with optimal expectations can exhibit intertemporal preference reversal, a greater readiness to accept commitment, regret, and a context effect in which non-chosen actions can affect utility.

Our model of beliefs differs markedly from treatments of risk in economics. While early models in macroeconomics specify beliefs exogenously as naive, myopic, or partially updated (e.g. Nerlove (1958)), since Muth (1960, 1961) and Lucas (1976) nearly all research has proceeded under the rational expectations assumption that subjective and objective beliefs coincide. There are two main arguments for this. First, the alternatives to rationality lack discipline. But our model provides exactly such discipline for subjective beliefs by specifying an objective for beliefs, that they maximize lifetime well-being. The second argument is that rational expectations is an optimal "as if" – agents have the incentive to hold rational beliefs (or act as if they do) because these expectations make the agent as well off as they can be. However, this rationale for rational expectations relies on an inconsistency: agents care about the future but at the same time expectations about the future do not affect current felicity. Our approach of optimal expectations is the outcome of an optimal "as if" argument that takes into account the fact that agents care in the present about utility flows that are expected in the future.

Most microeconomic models assume that agents share common prior beliefs. This "Harsanyi doctrine" is weaker than the assumption of rational expectations that all agents' prior beliefs are equal to the objective probabilities governing equilibrium dynamics. But like rational expectations, the common priors assumption is quite restrictive and does not allow agents to "agree to disagree" (Aumann (1976)). Savage (1954) provides axiomatic foundations for a more general theory in which agents hold arbitrary prior beliefs, so agents can agree to disagree. But if beliefs can be arbitrary, theory provides little structure or predictive power. Optimal expectations provides discipline to the study of subjective beliefs and heterogeneous priors. Framed in this way, optimal expectations is a theory of prior beliefs for Bayesian rational agents.

The key assumption that agents derive current felicity from expectations of future pleasures has its roots in the origins of utilitarianism. Detailed expositions on anticipatory utility can be found in the work of Bentham, Hume, Böhm-Barwerk and other early economists. More recently, the temporal elements of the utility concept have re-emerged in research at the juncture of psychology and economics (Loewenstein (1987), Kahneman, Wakker, and Sarin (1997), Kahneman (2000)), and have been incorporated formally into economic models in the form of belief-dependent utility by Geanakoplos, Pearce, and Stacchetti (1989), Caplin and Leahy (2001), and Yariv (2001). In particular, Caplin and Leahy (2000) shows that competitive equilibria are generically intertemporally sub-optimal and so opens the door for belief distortion to increase well-being.

Several papers in economics study related models in which forward-looking agents distort beliefs. In particular, Akerlof and Dickens (1982) models agents as choosing beliefs to minimize their discomfort from fear of bad outcomes. In a two-period model, agents with rational beliefs choose an industry to work in, understanding that in the second period they will distort their beliefs about the hazards of their work and perhaps not invest in safety technology. Landier (2000) studies a two-period game in which agents choose a prior before receiving a signal and subsequently taking an action based on their updated beliefs. Unlike our approach, belief dynamics are not Bayesian; common to our approach, agents tend to save less and be optimistic about portfolio returns. Finally, Bénabou and Tirole (2002) and Harbaugh (2002) analyze belief biases as resulting from conflicting multiple selves that play intra-personal games constrained by self-reputation. While our approach is not directly related to such settings, models of intra-personal games, bounded rationality, and incomplete memory suggest mechanisms for how households achieve optimal expectations in the face of possibly contradictory data.

Psychological theories provide many channels through which the human mind is able to hold beliefs inconsistent with the rational processing of objective data. First, individuals forget and remember events based on associativeness, rehearsal, and salience.[2] To the extent that these characteristics of memory bias the agent to remember better outcomes and to perceive them as more likely in the future, biases in beliefs are optimistic as in our optimal expectations framework. Second, most human behavior is not based on conscious cognition but is automatic, processed only in the limbic system and not the cortex (Bargh and Chartrand (1999)). If automatic processing is optimistic, then the agent may naturally approach problems with optimistic biases. However, the agent may also choose to apply cognition to discipline belief biases when the stakes are large, as in our optimal expectations framework.

The structure of the paper is as follows. In Section 2, we introduce the general optimal expectations framework. Subsequently, we use the optimal expectations framework to study behavior in three different canonical economic settings. Section 3 studies a two-period two-asset portfolio choice problem and shows that agents hold beliefs that are biased towards the belief that their investments will pay off well. Section 4 shows that in a two-agent economy of this type

[2] See for example Mullainathan (2002).

with no aggregate risk, optimal expectations are heterogeneous and agents gamble against one another. Section 5 analyzes a classical consumption-savings problem of an agent with quadratic utility receiving stochastic labor income over time and shows that the agent is biased towards optimism and is overconfident, and so saves less than a rational agent. Section 6 concludes. An appendix contains proofs of all propositions.

2 The optimal expectations framework

To solve for optimal beliefs and the resulting actions of agents, we proceed in two steps. First, we describe the problem of the agent given an arbitrary set of beliefs. At any point in time, agents maximize felicity, the present discounted value of expected flow utilities. Second, optimal expectations are the set of beliefs that maximize lifetime well-being in the initial period. Lifetime well-being is the expected discounted sum of the agent's felicities at each point in time, and so is a function of the agent's beliefs and the actions these beliefs induce.

2.1 Optimization given beliefs

Consider a wide and canonical class of optimization problems. In each period from 1 to T, agents take their beliefs as given and choose a vector of control variables, c_t, and the implied evolution of a vector of state variables, x_t, to maximize their happiness. We consider first a world where the uncertainty can be described by a finite number of states, S.[3] Let $\pi\left(s_t|\underline{s}_{t-1}\right)$ denote the true probability that state $s_t \in S$ is realized after state history $\underline{s}_{t-1} \in \underline{S}_{t-1}$. We depart from the canonical model in that agents are endowed with subjective probabilities that may not coincide with objective probabilities. Conditional and unconditional subjective probabilities are denoted by $\hat{\pi}\left(s_t|\underline{s}_{t-1}\right)$ and $\hat{\pi}\left(\underline{s}_t\right)$ respectively, and satisfy the basic properties of probabilities (precisely specified subsequently).

The felicity of an agent at time t depends on current utility flows and subjective expected future utility flows,

$$V\left(x_t;\underline{s}_t,\{\hat{\pi}\}\right) = \hat{E}_t \sum_{\tau=0}^{T-t} \beta^{\tau} u\left(x_{t+\tau}, c_{t+\tau}\right), \tag{1}$$

[3] Appendix A defines optimal expectations for the situation with a continuous state space.

where $\hat{E}_t$ is the subjective expectations operator, $0 < \beta < 1$, and $u(x_{t+\tau}, \cdot)$ is the flow utility function which is increasing and concave in $c_{t+\tau}$. Crucially, felicity depends on the complete set of subjective conditional beliefs, denoted $\{\hat{\pi}\}$. If β were zero, there would be no forward-looking behavior. While it is not crucial for our analysis, we allow the possibility that felicity additionally depends on past utility flows, which we capture by the function $M\left(\underline{\mathrm{x}}_{t-1}, \underline{\mathrm{c}}_{t-1}\right) = \sum_{\tau=1}^{t-1} \delta^{\tau} u\left(x_{t-\tau}, c_{t-\tau}\right)$ where $0 \leq \delta \leq \beta^{-1}$ and $\underline{\mathrm{x}}_{t-1}$ and $\underline{\mathrm{c}}_{t-1}$ denote the histories of x and c respectively. In the special case of $\delta = 0$ the agent has no experienced utility and current felicity collapses to equation (1).

The agent's problem is standard: at each time t, the agent chooses control variables to maximize felicity subject to the evolution of the state variables, the initial level of x_t and terminal conditions on x_{T+1}. Formally,

$$V^*\left(x_t; \underline{\mathrm{s}}_t, \{\hat{\pi}\}\right) = \max_{\{c_t\}} \ \hat{E}_t \sum_{\tau=0}^{T-t} \beta^{\tau} u\left(x_{t+\tau}, c_{t+\tau}\right)$$

subject to

$$\begin{aligned} x_{t+1} &= g\left(x_t, c_t, s_{t+1}\right) & (2) \\ h\left(x_{T+1}\right) &\geq 0 & (3) \end{aligned}$$

where the agent takes x_t as given, $h(\cdot)$ gives the endpoint condition, and $g(\cdot)$ gives the evolution of the state variable and is continuous and differentiable in x and c.

Denoting the optimal choices of the control as $\{c^*\left(\underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)\}$, the induced path of the state variable as $\{x^*\left(\underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)\}$, and the corresponding histories as $\underline{\mathrm{c}}^*\left(\underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)$ and $\underline{\mathrm{x}}^*\left(\underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)$, we can write V^* and M^* in a recursive formulation as:

$$\begin{aligned} V^*\left(x_t; \underline{\mathrm{s}}_t, \{\hat{\pi}\}\right) &= \max_{c_t} \ u\left(x_t, c_t\right) + \beta \sum_{s_{t+1} \in S} \hat{\pi}\left(s_{t+1} | \underline{\mathrm{s}}_t\right) V^*\left(g\left(x_t, c_t, s_{t+1}\right); s_{t+1}, \underline{\mathrm{s}}_t, \{\hat{\pi}\}\right) & (4) \\ & \quad \text{subject to (2) and (3)} \\ M\left(\underline{\mathrm{x}}^*_{t-1}, \underline{\mathrm{c}}^*_{t-1}\right) &= \delta u\left(x^*_{t-1}, c^*_{t-1}\right) + \delta M\left(\underline{\mathrm{x}}^*_{t-2}, \underline{\mathrm{c}}^*_{t-2}\right) \\ M_1 &= 0, \quad V_{T+1} = 0 \end{aligned}$$

Where not ambiguous, we collapse notation as c^*_t for $c^*\left(\underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)$, V^*_t for $V^*\left(x_t; \underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)$ and V^{**}_t for felicity along the optimal path, $V^*\left(x^*_t; \underline{\mathrm{s}}_t, \{\hat{\pi}\}\right)$.

So far we have focused on the optimization problem of a single agent. In a competitive economy, each agent faces this maximization problem taking as given his beliefs and the stochastic process of payoff-relevant aggregate variables, and markets clear. Beliefs may differ across agents. Specifically, x_t^i includes the payoff-relevant variables that the agent takes as given, and so reflects the actions of all other agents in the economy. The equilibrium choice of control variables for each agent implies an equilibrium allocation $\{\underline{x}_T, \underline{c}_T\}$, where $\{\underline{c}_T\}$ without a superscript i denotes *all agents*' control variable *along any possible path* of the event tree. In sum, the problem remains standard, with the exception that agents' prior beliefs may be heterogeneous.

2.2 Optimal beliefs

Subjective beliefs are a complete set of conditional probabilities for each branch after any history of the event tree, $\left\{\hat{\pi}\left(s_t|\underline{s}_{t-1}\right)\right\}$. We require that subjective probabilities satisfy four properties.

Assumption 1 (Restrictions on probabilities)

(i) $\sum_{s_t \in S} \hat{\pi}\left(s_t|\underline{s}_{t-1}\right) = 1$

(ii) $\hat{\pi}\left(s_t|\underline{s}_{t-1}\right) \geq 0$

(iii) $\hat{\pi}\left(\underline{s}'_t\right) = \hat{\pi}\left(s'_t|\underline{s}'_{t-1}\right) \hat{\pi}\left(s'_{t-1}|\underline{s}'_{t-2}\right) \cdots \hat{\pi}\left(s'_1\right)$

(iv) $\hat{\pi}\left(s_t|\underline{s}_{t-1}\right) = 0$ if $\pi\left(s_t|\underline{s}_{t-1}\right) = 0$.

Assumption 1(i) is simply that probabilities sum to one. Assumptions 1(i) – (iii) imply that the law of iterated expectations holds for subjective probabilities. Assumption 1(iv) implies that in order to believe that something is possible, it must be possible. You cannot believe you will win the lottery unless you buy a lottery ticket.

We further consider the class of problems for which a solution exists and for which V_t^{**} and M_t^{**} are less than infinite. While the conditions to ensure this are standard, we require that these properties hold for all possible subjective beliefs.

Assumption 2 (Conditions on agent's problem)

(i) $E\left[V^*\left(x^*\left(\underline{s}_t, \{\hat{\pi}\}\right); \underline{s}_t, \{\hat{\pi}\}\right)\right] < \infty$ for all $\underline{s}_t$ and for all $\{\hat{\pi}\}$ satisfying Assumption 1

(ii) $E\left[M\left(x^*\left(\underline{s}_{t-1}, \{\hat{\pi}\}\right), c^*\left(\underline{s}_{t-1}, \{\hat{\pi}\}\right)\right)\right] < \infty$ for all $\underline{s}_t$ and for all $\{\hat{\pi}\}$ satisfying Assumption 1.

Optimal expectations are the subjective probabilities that maximize expected total or lifetime well-being, $\mathcal{W}$, given that agents are optimizing.[4] Following Caplin and Leahy (2000), we define lifetime well-being as the discounted sum of felicity of the agent over its life, $E\left[\sum_{t=1}^{T} \beta^{t-1} \left(M_t + V_t\right)\right]$. Given agent optimization, felicity in period t is $M_t^{**} + V_t^{**}$, so that beliefs impact felicity directly through anticipation of future flow utility in V_t^{**} and indirectly through their effects on agent behavior, x_t^*.

Definition 1 *Optimal expectations (OE) are a set of subjective probabilities* $\left\{\hat{\pi}^{OE}\left(s_t|\underline{s}_{t-1}\right)\right\}$ *that maximize*

$$\mathcal{W}^* := E\left[\sum_{t=1}^{T} \beta^{t-1} \left(M\left(\underline{x}^*\left(\underline{s}_{t-1}, \{\hat{\pi}\}\right), \underline{c}^*\left(\underline{s}_{t-1}, \{\hat{\pi}\}\right)\right) + V^*\left(x^*\left(\underline{s}_t, \{\hat{\pi}\}\right); \underline{s}_t, \{\hat{\pi}\}\right)\right)\right] \qquad (5)$$

subject to the four restrictions on subjective probabilities (Assumption 1).

Optimal expectations exist if $c^{OE}\left(\underline{s}_t\right)$ and $x^{OE}\left(\underline{s}_t\right)$ are continuous in probabilities $\hat{\pi}\left(s_t|\underline{s}_{t-1}\right)$ that satisfy Assumption 1 for all t and $\underline{s}_{t-1}$, where for notational simplicity $c^{OE}\left(\underline{s}_t\right) := c^*\left(\underline{s}_t, \left\{\hat{\pi}^{OE}\right\}\right)$ and $x^{OE}\left(\underline{s}_t\right) := x^*\left(\underline{s}_t, \left\{\hat{\pi}^{OE}\right\}\right)$. This follows from the continuity of V_t^{**} and M_t^{**} in probabilities and controls, Assumption 2, and the compactness of probability spaces. For less regular problems, as for rational expectations equilibria, optimal expectations may or may not exist. As to uniqueness, optimal beliefs need not be unique, as will be clear from the subsequent use of this concept.

In an economy with multiple agents, each agent's beliefs maximize equation (5), where the state variables x^i and control variables c^i are indexed by i, taking the beliefs and actions of the other agents as given.

Definition 2 *A competitive optimal expectations equilibrium (OEE) is a set of beliefs for each agent and an allocation such that*
(i) each agent has optimal expectations, taking as given the stochastic process for aggregate variables;
(ii) each agent solves equation (4) at each t*, taking as given his beliefs and the stochastic process*

[4] A useful alternative for infinite horizon problems is that beliefs maximize average instead of total felicity.

for aggregate variables;

(iii) markets clear.

Intuitively, an optimal expectations equilibrium (OEE) consists of a set of beliefs for each agent i and the corresponding equilibrium allocations induced by optimization given these beliefs. The optimal beliefs of each agent take as given the aggregate dynamics, and the optimal actions take as given the perceived aggregate dynamics.

2.3 Discussion

Before proceeding to the application of optimal expectations, it is worth emphasizing several points.

First, because probabilities, $\hat{\pi}^{OE}\left(s_t|\underline{s}_{t-1}\right)$, are chosen once and forever, the law of iterated expectations holds with respect to the subjective probability measure and standard dynamic programming can be used to solve the agent's optimization problem. An alternative interpretation of optimal conditional probabilities is instead that the agent is endowed with optimal priors over the state space, $\hat{\pi}^{OE}\left(\underline{s}_T\right)$, and learns and updates over time according to Bayes' rule.[5] Thus agents are completely "Bayesian" rational given what they know about the economic environment.

Second, optimal expectations are those that maximize lifetime well-being. The argument that is traditionally made for the assumption of rational beliefs – that such beliefs lead agents to the best outcomes – is correct only if one assumes that expected future utility flows do not affect present felicity. This is a somewhat schizophrenic view: one part of the agent makes plans that trade off present and expected future utility flows, while another part of the agent actually enjoys utils but only from present consumption.[6] Under the Jevonian view that an agent who cares about the future has felicity that depends on expectations about the future, optimal expectations give agents the highest lifetime utility levels.

[5] The interpretation of the problem in terms of optimal priors requires that one specify agent beliefs following zero subjective probability events, situations in which Bayes rule provides no restrictions.

[6] See Loewenstein (1987) and the discussion of the Samuelsonian and Jevonian views of utility in Caplin and Leahy (2000).

To recast this point, we can ask what objective function for beliefs would make rational expectations optimal. This is the case if the objective function for beliefs omits anticipatory or memory utility, so that $\mathcal{W}^* = E\sum_{t=1}^{T}\beta^{t-1}u(x_t^*, c_t^*)$. This implies that either agents do not care about the future (V^* does not contain future u) or beliefs do not maximize lifetime well-being ($\mathcal{W}^*$ is defined over u^* instead of V^{**}). Because we view this as implausible, we restrict attention to objective functions for beliefs defined over V^{**}.

Third, this discussion also makes clear why lifetime well-being, $\mathcal{W}^*$, uses the objective expectations operator. Optimal beliefs are not those that maximize the agent's happiness only in the states that the agent views as most likely. Instead, optimal beliefs maximize the happiness of the agent on average, across repeated realizations of uncertainty. The objective expectation captures this since the actual unfolding of uncertainty over the agent's life is determined by objective probabilities.

Fourth, in all the examples of this paper, the key reason for belief distortion is that current felicity depends on anticipated future utility flows. We ensure this by studying situations in which the objective function for beliefs evaluated for rational beliefs is identical to the objective function of the agent. This is the case when $\delta = \beta^{-1}$, which we refer to as *preference consistency.* With preference consistency, lifetime well-being is maximized by the actions of an agent with rational expectations. Without preference consistency, there may be an additional incentive for the distortion of beliefs since an agent's ranking of utility flows across periods is not time-invariant (Caplin and Leahy (2000)). This disagreement over optimal actions in different periods would give an additional incentive for beliefs to be distorted to increase lifetime well-being.

That said, for most of our results we do not need to specify the value of δ. In particular, this allows for the case of preference consistency and the case in which $\delta = 0$. For the latter case, the analysis simplifies since all "experience utility" terms M_t vanish.

Fifth, optimal expectations could be derived from generalized objective functions. In particular, an earlier version of this paper considered the possibility that the rate at which future felicity is discounted in the objective function for beliefs differed from the rate at which the agent discounted future utility flows.

Sixth, optimal subjective probabilities are chosen without any direct relation to reality. This frictionless world provides insight into the behaviors generated by the incentive to look forward with optimism when belief distortion is limited by the costs of poor outcomes. In fact, it may be that beliefs cannot be distorted far from reality for additional reasons. At some cost in terms of tractability, the frictionless model could be extended to include constraints that penalize larger distortions from reality. Beliefs would then bear some relation to reality even in circumstances in which there are no costs associated with behavior caused by distorted beliefs.

Seventh, this model is also extreme in that $\hat{\pi}^{OE}\left(s_t|\underline{s}_{t-1}\right)$ are assigned at every node, so that belief distortion can vary significantly across states with similar outcomes. Again at some cost in terms of tractability, one could require that belief distortions be restricted to be "smooth" or lie on a coarser partition of the probability space. For example, one might require that belief biases be similar for states of the world that lead to similar payouts for a given asset. Alternatively, one could restrict the set that optimal beliefs are chosen over to be a set of parsimonious models of the environment. For example, we might require that the agent believe that their income process is some first-order Markov process rather than allow belief distortions to be completely history dependent. While such restrictions are somewhat appealing, it is difficult to know how to discipline the choice of such restrictions.[7]

Finally, one might be concerned that agents with optimal expectations might be driven to extinction by agents with rational beliefs. But evolutionary arguments need not favor rational expectations. Since optimal expectations respond to the costs of mistakes, agents with optimal expectations are hard to exploit. And many economic environments favor agents who take on more risk (DeLong, Shleifer, Summers, and Waldmann (1990)). Finally, consistent with our choice of $\mathcal{W}^*$, there is a biological link between happiness and better health (Taylor and Brown (1988)).

[7] If the agent were aware that his prior/model is chosen from a set of parsimonious models, then he might questions these beliefs. In this case, it would make sense to impose the additional restriction that only priors can be chosen for which the agent cannot detect the misspecification, an approach being pursued in the literature on robust control. By not restricting the choice set over priors we avoid these complications.

3 Portfolio choice: optimism and gambling

In this section we consider a two-period investment problem in which an agent chooses between assets in the first period of life and consumes the payoff of the portfolio in the second period of life. We show that agents are optimistic about the payout of their own investments and do not hold perfectly diversified portfolios. This matches stylized facts about investor behavior. The subsequent section places two of these agents into a general equilibrium model with no aggregate risk, and shows that these agents disagree about the returns of assets.

3.1 Portfolio choice given beliefs

There are two periods and two assets. In period one, the agent allocates his unit endowment between a risk-free asset with gross return R and a risky asset with gross return $R+Z$ (Z is the excess return of the risky asset over the risk-free rate). In period two, the agent consumes the payoff from his first-period investment.

The agent chooses his portfolio share to invest in the risky asset, w, to maximize expected utility:

$$\begin{aligned} V_1^* = \max_{w} \quad & \beta \hat{E}_1 \left[u\left(c\right) \right] \\ \text{s.t.} \quad & c = R + wZ \\ & c \geq 0 \text{ in all states} \end{aligned}$$

where $u(\cdot)$ is the utility function over consumption, $u' > 0$, $u'' < 0$ and satisfying the Inada conditions. The second constraint is set by the market. Since consumption cannot be negative, the constraint follows from the market requiring the agent to be able to meet his payment obligations in all future states.

Uncertainty is characterized by S states, with ex post excess return Z_s and probabilities $\pi_s > 0$ for $s = 1, \ldots, S$. Let the states be ordered so that the larger the state, the larger the payoff, $Z_{s+1} > Z_s$, and $-R < Z_1 < 0 < Z_S < R$, $Z_s \neq Z_{s'}$ for $s \neq s'$. Beliefs are given by $\{\hat{\pi}_s\}_{s=1}^{S}$ satisfying Assumption 1.

Noting that the second constraint can only bind for the highest or lowest payoff state, the agent problem can be written as a Lagrangian with multipliers λ_1 and λ_S,

$$\max_w \beta \sum_{s=1}^{S} \hat{\pi}_s u\left(R + wZ_s\right) - \lambda_1 \left(R + wZ_1\right) - \lambda_S \left(R + wZ_S\right).$$

The necessary conditions for an optimal w are

$$\begin{aligned} 0 &= \sum_{s=1}^{S} \hat{\pi}_s u'\left(R + w^* Z_s\right) Z_s - \lambda_1 Z_1 - \lambda_S Z_S, \\ 0 &= \lambda_1 \left(R + w^* Z_1\right), \\ 0 &= \lambda_S \left(R + w^* Z_S\right). \end{aligned}$$

It turns out that optimal beliefs never lead the agent to choose $R + w^* Z_s = 0$. To see this, suppose that $R + w^* Z_s = 0$ for some s and consider an infinitesimal change in probability that results in an increase of consumption in this state. By the Inada condition $u'(0) = \infty$, this causes an infinite marginal increase in lifetime well-being. Thus, optimal expectations imply $R + w^* Z_s \neq 0$ for any s. By complementary slackness, $\lambda_s = 0$ for all s, and the optimal portfolio is uniquely determined by

$$0 = \sum_{s=1}^{S} \hat{\pi}_s u'\left(R + w^* Z_s\right) Z_s \;\Rightarrow w^*\left(\{\hat{\pi}\}\right). \tag{6}$$

3.2 Optimal beliefs

Optimal beliefs are a set of $\hat{\pi}_s$ for $s = 1, \ldots, S-1$ with $\hat{\pi}_S = 1 - \sum_{s=1}^{S-1} \hat{\pi}_s$ that maximize total well-being, the expected discounted sum of felicities in periods 1 and 2. In period 1, the agent's felicity is the subjectively expected (anticipated) utility flow in the future period, discounted by β; in period 2, the agent's felicity is the utility flow from actual consumption.

$$\begin{aligned} &\max_{\hat{\pi}} \quad E\left[V_1^{**} + \beta V_2^{**}\right] \\ &\max_{\hat{\pi}} \quad E\left[\beta \hat{E}_1\left[u(c)\right] + \beta u(c)\right] \\ &\max_{\hat{\pi}} \quad \beta \sum_{s=1}^{S} \hat{\pi}_s u\left(R + w^*\left(\{\hat{\pi}\}\right) Z_s\right) + \beta \sum_{s=1}^{S} \pi_s u\left(R + w^*\left(\{\hat{\pi}\}\right) Z_s\right) \end{aligned}$$

where $w^*\left(\{\hat{\pi}\}\right)$ is given implicitly by equation (6). The first-order conditions for the choice of $\hat{\pi}_s$ are

$$0 = \beta u_{s'} - \beta u_S + \beta \sum_{s=1}^{S} \hat{\pi}_s^{OE} u'\left(R + w^* Z_s\right) Z_s \frac{dw^*}{d\hat{\pi}_{s'}} + \beta \sum_{s=1}^{S} \pi_s u'\left(R + w^* Z_s\right) Z_s \frac{dw^*}{d\hat{\pi}_{s'}}.$$

By the envelope condition, small changes in portfolio choice from the optimum caused by small changes in subjective probabilities lead to no change in expected utility, so that this condition simplifies to

$$\beta \left(u_S - u_{s'}\right) = \beta \sum_{s=1}^{S} \pi_s u' \left(R + w^* Z_s\right) Z_s \frac{dw^*}{d\hat{\pi}_{s'}}. \tag{7}$$

The left-hand side is the marginal gain in 'dream utility' at t from increasing $\hat{\pi}_S$ at the expense of $\hat{\pi}_{s'}$ and is always positive; the right-hand side is the marginal loss in expected utility in $t+1$ from the resultant change in the portfolio share of the risky asset. In equilibrium, the gain in dream utility balances the costs of distorting actual behavior.

Let w^{RE} denote the optimal portfolio choice for rational beliefs. The following proposition, proved in the appendix, states that the agent with optimal expectations is optimistic about the payout of his portfolio. Further, the agent with optimal expectations either takes an opposite position relative to the agent with rational beliefs or is more aggressive – investing even more if the rational agent invests, or shorting more if the rational agent shorts.

Proposition 1 *(Excess risk taking due to optimism)*
(i) if $w^{OE} > 0$, $\sum_{s=1}^{S} \left(\hat{\pi}_s - \pi_s\right) u' \left(R + w^{OE} Z_s\right) Z_s > 0$*; if* $w^{OE} < 0$, $\sum_{s=1}^{S} \left(\hat{\pi}_s - \pi_s\right) u' \left(R + w^{OE} Z_s\right) Z_s < 0$.
(ii) if $E[Z] > 0$*, then* $w^{RE} > 0$*,* $w^{OE} > w^{RE}$ *or* $w^{OE} < 0$*; if* $E[Z] < 0$*, then* $w^{RE} < 0$*,* $w^{OE} < w^{RE}$ *or* $w^{OE} > 0$*; if* $E[Z] = 0$*, then* $w^{RE} = 0$ *and* $w^{OE} \neq 0$.

The first part of the proposition states that agents with optimal expectations on average hold beliefs that are biased upward for states in which their chosen portfolio payout is high and biased downward for states in which their portfolio payout is low. To see this, note that $u'_s > 0$ for all s, and Z_s is positive for large s and negative for small s. For $w^{OE} > 0$, optimal expectations on average bias up the subjective probability for larger or positive excess return states at the expense of smaller or negative excess return states.

The second part of the proposition characterizes behavior. Consider first the situation when $E[Z] = 0$. The rational agent chooses $w^{RE} = 0$ since the expected return on the risky asset is the same as the risk-free asset and the risky asset is risky. But a deviation of beliefs from rational beliefs can increase lifetime well-being. A small deviation leads the agent to choose $w \neq 0$ which implies that consumption is no longer perfectly smoothed across future states. The agent now believes that he is holding (shorting) an asset with positive (negative) expected payoff. The costs of imperfect consumption smoothing are dominated by the gain in anticipated future utility. Thus $\hat{\pi}^{OE} \neq \pi$ and $w^{OE} \neq w^{RE} = 0$. An implication is that, from the perspective of objective probabilities, agents with optimal expectations are underdiversified. That is, relative to w^{OE}, a portfolio with the same objective expected return and less objective risk is available, since $E\left[R + w^{OE} Z_s\right] = E\left[R + w^{RE} Z_s\right]$ but $Var\left[R + w^{OE} Z_s\right] > Var\left[R + w^{RE} Z_s\right]$.

For $E[Z] > 0$, the second part of the proposition states that the household either invests more than the rational agent in the risky asset or shorts the risky asset, and vice versa for $E[Z] < 0$. Why would the agent take a position in the opposite direction to the rational agent, when this implies that he is taking a negative expected payoff gamble? This occurs when anticipatory utility in the contrarian position is sufficiently large. For many utility functions, this is the case when the asset has the properties similar to a lottery ticket, that is when the asset is skewed in the opposite direction of the mean payoff.

To illustrate this point, consider a world with two states and an asset with negative expected excess payoff, $E[Z] =: \mu_Z < 0$. We specify the payoffs Z_1 and Z_2, such that, as we vary probabilities the mean and variance, σ_Z^2, stay constant, but skewness increases in π_1.

State	Probability	Excess Payoff
1	π_1	$Z_1 = \mu_Z - \sigma_Z \sqrt{\frac{1-\pi_1}{\pi_1}}$
2	$1 - \pi_1$	$Z_2 = \mu_Z + \sigma_Z \sqrt{\frac{\pi_1}{1-\pi_1}}$

When π_1 is large, the asset is similar to a lottery: the asset yields a small negative return with high probability and a large positive return with low probability.

Proposition 2 *For unbounded utility functions, there exists a* $\underline{\pi}$ *such that for all* $1 > \pi_1 > \underline{\pi}$ *(i)* $\hat{\pi}_1 < \pi_1$ *and (ii)* $w^{OE} > 0$ *even though* $E[Z] < 0$.

For the agent shorting the asset, when π_1 is close to unity, $\hat{\pi}_1 - \pi_1$ is near zero – subjective beliefs are necessarily near rational beliefs – and $w^*(\{\hat{\pi}\})$ is near w^{RE}. However, in this case, if the agent instead is optimistic about the payoff of the risky asset, $\hat{\pi}_1 < \pi_1$, then he can invest in the asset and dream about the asset paying off well. In fact, for π_1 near unity, $\hat{\pi}_1^{OE} < \pi_1$ and w^{OE} is positive. This type of behavior – buying stochastic assets with negative expected return and positive skewness – is widely observed in gambling and betting.

4 General equilibrium: endogenous heterogenous beliefs

In this section, we consider an exchange economy with no aggregate risk and derive endogenous heterogeneous beliefs. Further, in our model agents choose to hold idiosyncratic risk and gamble against one another in equilibrium when perfect consumption insurance is possible. These features match stylized facts about asset markets. People disagree about asset returns, gamble, and do not perfectly insure consumption. Finally, the price of the risky asset may differ from that in an economy populated by agents with rational beliefs.

The economy consists of two agents with the same characteristics and facing the same investment problem as in the previous section. There are two states in the second period and two assets (or "trees"), denoted b (bonds) and e (equity). Asset b pays 1 unit of consumption in both states; we normalize the return on this tree to 1 (R equals unity). Asset e returns $1 + Z$ and the ex post return on asset e is $1 + Z_s = \frac{1+\varepsilon_s}{P}$ where s indexes states. We assume $-1 < \varepsilon_1 < \varepsilon_2$. Agent i is initially endowed with X_b^i bonds and X_e^i equity shares. There is no aggregate uncertainty so that asset e is in zero net supply. Aggregate consumption in each state is thus the same, $\sum_{i=1}^2 X_b^i = X_b = C_1 = C_2$.

Agent i's problem is to take his beliefs, $\{\hat{\pi}_s^i\}$, and the price of equity, P, as given and choose her portfolio to maximize expected utility,

$$\max_w \beta \sum_s \hat{\pi}_s^i u\left(A^i\left(1 + w^i Z\right)\right)$$

given initial wealth

$$A^i = X_b^i + P X_e^i.$$

The first-order conditions for portfolio choice are

$$0 = \sum_s \hat{\pi}_s^i u'\left(c_s^{*i}\right)\left((1 + \varepsilon_s) - P\right).$$

In aggregate, beliefs maximize the lifetime well-being for each agent

$$\max_{\hat{\pi}} \sum_s \left[\beta \hat{\pi}_s^i u\left(c_s^{*i}\right) + \beta \pi_s u\left(c_s^{*i}\right)\right]$$

subject to the restrictions on probabilities (Assumption 1), the budget constraint (the definition of consumption), and the agent's first-order conditions for portfolio choice. The Lagrangian for each agent is

$$\begin{aligned} \mathcal{L} &= \sum_s \left[\beta \hat{\pi}_s^i u\left(\left(X_b^i + P X_e^i\right)\left(1 + w^{*i}\left(\tfrac{1+\varepsilon_s}{P_e} - 1\right)\right)\right) + \beta \pi_s u\left(\left(X_b^i + P X_e^i\right)\left(1 + w^{*i}\left(\tfrac{1+\varepsilon_s}{P} - 1\right)\right)\right)\right] \\ &\quad -\lambda^i \left[\sum_s \hat{\pi}_s^i - 1\right] - \mu^i \left[\sum_s \hat{\pi}_s^i u'\left(\left(X_b^i + P X_e^i\right)\left(1 + w^{*i}\left(\frac{1+\varepsilon_s}{P} - 1\right)\right)\right)\left((1+\varepsilon_s) - P\right)\right] \end{aligned}$$

The equilibrium for the economy as a whole is characterized by the first-order conditions for each agent

$$\begin{aligned} w^i \quad &: \quad 0 = \sum_s \left[\left(\beta \hat{\pi}_s^{OE,i} + \beta \pi_s\right) u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P^{OE}} - 1\right)\right] \\ &\qquad -\mu^i \left[\sum_s \hat{\pi}_s^{OE,i} u''\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P^{OE}} - 1\right)^2 P^{OE}\right] \text{ for } i = 1,2 \end{aligned} \tag{8a}$$

$$\hat{\pi}_s^i \quad : \quad 0 = \beta u\left(c_s^{OE,i}\right) - \lambda^i - \mu^i \left[u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P^{OE}} - 1\right) P^{OE}\right] \text{ for } i = 1,2, \;\; s = 1,2 \tag{8b}$$

$$\lambda^{i:} \quad : \quad 0 = \lambda^i \left[\sum_s \hat{\pi}_s^{OE,i} - 1\right] \text{ for } i = 1,2$$

$$\mu^i \quad : \quad 0 = \mu^i \left[\sum_s \hat{\pi}_s^{OE,i} u'\left(c_s^{OE,i}\right)\left((1+\varepsilon_s) - P^{OE}\right)\right] \text{ for } i = 1,2 \tag{8c}$$

and the aggregate resource constraint

$$-P^{OE} \sum_i X_e^i w^{OE,i} = \sum_i X_b^i w^{OE,i}, \tag{9}$$

where $c_s^{OE,i} = \left(X_b^i + P^{OE} X_e^i\right)\left(1 + w^{OE,i}\left(\frac{1+\varepsilon_s}{P^{OE}} - 1\right)\right)$.

Before characterizing the equilibrium, we define gambling. Let x_e^i be agent i's equilibrium holding of equity, $x_e^i = w^i \frac{\left(X_b^i + P X_e^i\right)}{P}$. Agents are gambling against each other in an equilibrium $(*)$ if

$$x_e^{*i} \neq x_e^{RE,i} \text{ for } i = 1, \, 2.$$

This says that the amount of equity held by each agent is different in the $(*)$ equilibrium than in the rational expectations equilibrium.

Note that gambling is not implied by a difference between the equilibrium price of equity under rational and optimal expectations, $P^{OE} \neq P^{RE}$. Gambling results from disagreement about probabilities. Put differently, at the optimal expectations equilibrium we have for each agent

$$\sum_s \hat{\pi}_s^{OE,i} u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P^{OE}}-1\right)=0.$$

If agents are not gambling, then

$$\sum_s \pi_s u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P^{RE}}-1\right)=0.$$

Thus if optimal beliefs differ from objective beliefs, the price of equity will differ in the rational expectations and optimal expectations equilibria even without gambling. Mispricing does not imply gambling. We now state our proposition.

Proposition 3 *(Gambling without aggregate risk)*
(i) $w^{RE,i}=0$, $x_e^{RE,i}=0$,
(ii) agents gamble, $x_e^{OE,1} \neq x_e^{RE,i} \neq x_e^{OE,2}$,
(iii) in any equilibrium $\hat{\pi}_1^{OE,i} > \pi_1$, $\hat{\pi}_2^{OE,i} < \pi_2$, $w^{OE,i} < 0$, $c_1^{OE,i} > c_2^{OE,i}$, *and* $\hat{\pi}_2^{OE,-i} > \pi_2$, $\hat{\pi}_1^{OE,-i} < \pi_1$, $w^{OE,-i} > 0$, $c_2^{OE,-i} > c_1^{OE,-i}$ *for* $i = 1$ *or* 2.

The first point states that agents in the rational expectations equilibrium perfectly insure their consumption by trading so that neither agent holds the risky asset. The second point follows from the same logic as in the partial equilibrium section. Agents have a loss from a small amount of betting against one another, which is dominated by the gain in dream utility from gambling. The final point emphasizes the source of the gambling. Each agent believes a different state is more likely than it really is, and they trade so that each of them realizes higher consumption in the state they view as unrealistically likely. Thus, a feature of the optimal expectations equilibrium is endogenous heterogeneity in beliefs.

While this model implies multiple equilibrium, it is natural to think that agents choose to be optimistic about the state in which their initial endowment pays off more. This equilibrium minimizes trading costs. This is consistent with people believing in the economic performance of their own companies or countries. Recent studies of pension behavior find investors choosing to hold a larger share of their wealth in the equity of their employer than rational models suggest is optimal. Similarly, investors have a home bias, they hold a larger share of their wealth in the equity of their own country than rational models suggest is optimal.[8]

5 Consumption and saving over time: undersaving and overconfidence

This section considers the behavior of an agent with optimal expectations in a multi-period consumption-saving problem with stochastic income. We show that the agent with quadratic utility overestimates the mean of future income and underestimates the uncertainty associated with future income. That is, the agent is both unrealistically optimistic and overconfident. This is consistent with survey evidence that shows that growth rate of expected consumption is greater than that of actual consumption.

5.1 Consumption given beliefs

In each period $t = 1, \ldots, T$, the agent chooses consumption and saving to maximize the expected present discounted value of utility flows from consumption subject to a budget constraint.

$$
\begin{aligned}
V^*\left(A_t; \underline{y}_t, \left\{\hat{\Pi}\right\}\right) &= \max_{\{c_t\}} \hat{E}\left[\sum_{\tau=0}^{T-t} \beta^\tau u\left(c_{t+\tau}\right) | \underline{y}_t\right] \\
\text{s.t.} \qquad & \sum_{\tau=0}^{T-t} R^{-\tau}\left(c_{t+\tau} - y_{t+\tau}\right) = A_t \\
u\left(c_{t+\tau}\right) &= a c_{t+\tau} - \frac{b}{2} c_{t+\tau}^2
\end{aligned}
$$

where initial wealth $A_1 = 0$, $a, b > 0$ and $\beta R = 1$. The only uncertainty is over income, y_t. y_t is continuously distributed, has support $\left[\underline{y}, \bar{y}\right] = Y$ where $0 < \underline{y} < \bar{y} < \frac{a}{bT}$, is independent over time so that $\Pi\left(y_t | \underline{y}_{t-1}\right) = \Pi\left(y_t\right)$, and $d\Pi\left(y_t\right) > 0$ for all $y \in Y$. Subjective probabilities are

[8] For surveys on overinvesting in the equity of one's company and one's country, see Poterba (2003) and Lewis (1999), respectively.

denoted by $\hat{\Pi}\left(y_t|\underline{y}_{t-1}\right)$ and do not have to be independently distributed over time. Appendix A states the restrictions on probabilities and objective functions for a continuous state space.

Assuming an interior solution, the necessary conditions for an optimum are, for $t = 1$ to $T-1$,

$$\begin{aligned} 0 &= u'\left(c_t^*\right) - \beta R \int_{y_{t+1}\in Y} \frac{dV^*\left(A_{t+1};\underline{y}_{t+1},\left\{\hat{\Pi}\right\}\right)}{dA_{t+1}} d\hat{\Pi}\left(y_{t+1}|\underline{y}_t\right) \\ \frac{dV^*\left(A_t;\underline{y}_t,\left\{\hat{\Pi}\right\}\right)}{dA_t} &= \beta R \int_{y_{t+1}\in Y} \frac{dV^*\left(A_{t+1};\underline{y}_{t+1},\left\{\hat{\Pi}\right\}\right)}{dA_{t+1}} d\hat{\Pi}\left(y_{t+1}|\underline{y}_t\right). \end{aligned}$$

Combining these conditions and the assumption of quadratic utility gives the Hall random walk result for consumption but for subjective beliefs

$$c_t^* = \hat{E}\left[c_{t+1}^*|\underline{y}_t\right]. \tag{10}$$

Substituting back into the budget constraint gives the optimal consumption rule

$$c_t^* = \tfrac{1-R^{-1}}{1-R^{-(T-t)}}\left(A_t + y_t + \sum_{\tau=1}^{T-t} R^{-\tau}\hat{E}\left[y_{t+\tau}|\underline{y}_t\right]\right). \tag{11}$$

Optimal consumption depends on subjective expectations of future income and the history of income realizations through A_t. Because quadratic utility exhibits certainty equivalence from the perspective of the agent, the problem simplifies significantly. Given the subjective expectation of future income, the subjective variance (and higher moments) of the income process are irrelevant for the optimal consumption-saving choices of the agent.

5.2 Optimal beliefs

We want to choose $\hat{\Pi}\left(y_t|\underline{y}_{t-1}\right)$ for all $\underline{y}_{t-1}$to maximize lifetime well-being subject to the probability conditions and the agent's optimal behavior given beliefs. With quadratic utility, the elements of the objective function for beliefs are

$$\begin{aligned} M_t^{**} &= \sum_{r=1}^{t-1}\delta^r\left(ac_{t-r}^* - \frac{b}{2}\left(c_{t-r}^*\right)^2\right) \\ V_t^{**} &= \sum_{r=0}^{T-t}\beta^r\hat{E}\left[ac_{t+r}^* - \frac{b}{2}\left(c_{t+r}^*\right)^2|\underline{y}_t\right]. \end{aligned}$$

Since the objective is concave in future consumption and since the agent's behavior depends only on the subjective certainty-equivalent of future income, optimal beliefs minimize subjective uncertainty. Thus, future income is optimally perceived as certain, which is an extreme form of overconfidence.

We incorporate optimal behavior directly into the value functions and characterize consumption choices implied by optimal beliefs, $\{c_t^{OE}\}$. Optimal beliefs, $\left\{\hat{\Pi}^{OE}\right\}$, implement these consumption choices given optimal behavior on the part of the agent. In taking this approach, we are assuming that the optimal choice of consumption and thus $\hat{E}\left[y_{t+\tau}|\underline{\mathrm{y}}_t\right]$ does not require violation of the assumptions on probability, which can be checked.[9]

To proceed, optimal behavior is summarized by the agent's Euler equation. Using the Euler equation and the fact that subjective certainty implies $\hat{E}\left[u\left(c_{t+\tau}^*\right)|\underline{\mathrm{y}}_t\right] = u\left(\hat{E}\left[c_{t+\tau}^*|\underline{\mathrm{y}}_t\right]\right)$, the value functions over future consumption become

$$\begin{aligned} V_t^{**} &= \sum_{r=0}^{T-t} \beta^r u\left(\hat{E}\left[c_{t+r}^*|\underline{\mathrm{y}}_t\right]\right) \\ &= u\left(c_t^*\right) \sum_{r=0}^{T-t} \beta^r. \end{aligned}$$

Subjective expectations are chosen to yield the path of $\{c_t^*\}$ that maximizes

$$\begin{aligned} & E\left[\begin{array}{l} \underbrace{u\left(c_1^*\right)\sum_{\tau=1}^{T}\beta^{\tau-1}}_{V_1^{**}} + \beta\underbrace{u\left(c_1^*\right)\delta}_{M_2^{**}} + \beta\underbrace{u\left(c_2^*\right)\sum_{\tau=1}^{T-1}\beta^{\tau-1}}_{V_2^{**}} + \beta^2\underbrace{\left(u\left(c_1^*\right)\delta^2 + u\left(c_2^*\right)\delta\right)}_{M_3^{**}} + \\ \qquad \beta^2\underbrace{u\left(c_3^*\right)\sum_{\tau=1}^{T-2}\beta^{\tau-1}}_{V_3^{**}} + \ldots + \beta^{T-1}\underbrace{\sum_{\tau=1}^{T-1}\delta^{T-\tau}u\left(c_\tau^*\right)}_{M_T^{**}} + \beta^{T-1}\underbrace{u\left(c_T^*\right)}_{V_T^{**}} \end{array}\right] \\ = \; & E\left[\sum_{t=1}^{T}\psi_t u\left(c_t^*\right)\right] \end{aligned} \tag{12}$$

subject to the budget constraint and where $\psi_t = \beta^{t-1}\left(1 + \sum_{\tau=1}^{T-t}\left(\beta^\tau + (\beta\delta)^\tau\right)\right)$. If there were no memory utility ($\delta = 0$), and the objective for beliefs ignored anticipatory utility, then $\psi_t = \beta^t$. In this case, the optimal consumption path is standard, and beliefs would be rational.

[9] If the support of y_t is small, belief distortion may be constrained by the range of possible income realizations. To incorporate these constraints directly, one instead replaces $c^*\left(\underline{\mathrm{y}}_t, \left\{\hat{\Pi}^{OE}\right\}\right)$ using equation (11) and searches for optimal $\hat{E}\left[y_{t+\tau}|\underline{\mathrm{y}}_t\right]$ while imposing the constraints imposed by Assumption 1 (iv).

Under optimal expectations, expected consumption growth is given by the first-order conditions

$$u'\left(c_t^{OE}\right) = \frac{\psi_{t+\tau}}{\psi_t} R^\tau \int_{\underline{\mathrm{y}}_{t+\tau}|\underline{\mathrm{y}}_t \in \times_\tau Y} u'\left(c_{t+\tau}^{OE}\right) d\Pi\left(\underline{\mathrm{y}}_{t+\tau}|\underline{\mathrm{y}}_t\right),$$

which implies that, for any sequence of income realizations,

$$c^{OE}\left(\underline{\mathrm{y}}_t\right) = \frac{a}{b} - \frac{\psi_{t+\tau}}{\psi_t} R^\tau \left(\frac{a}{b} - E\left[c^{OE}\left(\underline{\mathrm{y}}_{t+\tau}\right)|\underline{\mathrm{y}}_t\right]\right). \tag{13}$$

Level consumption is recovered by substituting into the budget constraint after taking objective expectations.

Given this characterization of optimal behavior, agents are optimistic at every time and state. Define human wealth as the present value of current and future labor income at t, $H_t = \sum_{\tau=0}^{T-t} R^{-\tau} y_{t+\tau}$.

Proposition 4 *(Overconsumption due to optimism)*
For all $t \in \{1, \ldots, T-1\}$,
(i) $\hat{E}\left[H_{t+1}|\underline{y}_t\right] > E\left[\hat{E}\left[H_{t+1}|\underline{y}_{t+1}\right]|\underline{y}_t\right]$;
(ii) $c_t^{OE} > E\left[c_{t+1}^{OE}|\underline{y}_t\right]$;
(iii) $\hat{E}\left[c_{t+1}^{OE}|\underline{y}_t\right] > E\left[c_{t+1}^{OE}|\underline{y}_t\right]$.

The first point of the proposition states that agents overestimate their present discounted value of labor income and on average revise their beliefs downward between t and $t+1$. This downward revision of expected lifetime wealth can come about directly due to y_{t+1} being on average less than expected, or due to news that the realized y_{t+1} brings on average about expectations of future income. The second point states that consumption on average falls between t and $t+1$. Because on average the agent revises down expected future income, on average consumption falls over time. The proof follows directly from the expected change in consumption given by equation (13) and noting that $\frac{a}{b} - c_t^{OE}\left(\underline{\mathrm{y}}_t\right) > 0$ and $\frac{\psi_{t+1}}{\psi_t} R < 1$. Finally, the optimal subjective expectation of future consumption exceeds the rational expectation of future consumption. This is optimism. Part (iii) follows from part (ii) and equation (10). In sum, households are unrealistically optimistic, and, in each period, are on average surprised that their incomes are lower than they expected, and so, on average, household consumption declines over time.

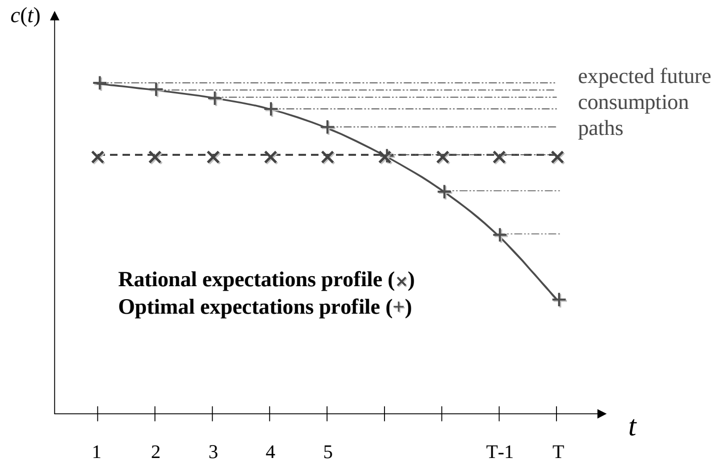

Figure 1: Average Life-cycle Consumption Profiles

Figure 1 summarizes these results. The agent starts life optimistic about future income. At each point in time the agent expects that on average consumption will remain at the same level. Over time, the agent learns on average that income is less than he expected, and consumption typically declines over the life.

This optimism matches survey evidence on desired and actual life-cycle consumption profiles. Barsky, Juster, Kimball, and Shapiro (1997) finds that households would choose upward sloping consumption profiles. But survey datasets on actual consumption reveal that households have downward sloping or flat consumption profiles (Gourinchas and Parker (2002), Attanasio (1999)). In our model, households expect and plan to have constant marginal utility since $\beta R - 1 = 0$. However, on average marginal utility rises at the age-specific rate $\frac{\psi_t}{\psi_{t+1}} - 1 > 0$. Thus, in the model, the desired rate of increase of consumption exceeds the average rate of increase, as in the real world. Also matching observed household consumption behavior, the model produces average life-cycle consumption profiles that are concave – consumption falls faster (or rises more slowly) later in life.

In general, in consumption-saving problems, the relative curvature of utility and marginal utility determine what beliefs are optimal. Uncertainty about the future enters the objective for beliefs both through the expected future *level* of utility and through the agent's behavior which depends on expected future *marginal* utility. For utility functions with decreasing absolute risk aversion, greater subjective uncertainty leads to greater precautionary saving through the curvature in marginal utility. This has some benefit in terms of less distortion of consumption. In such cases optimal beliefs may consist of a large positive bias for both expected income and its variance.

We conclude this section by using our consumption-saving problem to make three points about the dynamic choices of agents with optimal expectations.

First, given that in expectation the consumption of the agent is always declining, the costs of optimism early in life could be extreme for long-lived agents. But, illustrating a general point, optimal expectations depend on the horizon in a way that mitigate these possible costs. The behavior of an agent with a long horizon is close to that of an agent with rational expectations. For T large but finite, an agent with optimal expectations consumes a small amount more for most his life, leading to a significant decline in consumption at the end of life. As the horizon becomes infinite, at any fixed age, the agent with optimal beliefs chooses a level of consumption arbitrarily close to that of the agent with rational expectations and the subjective expectation of future labor income become nearly rational. Formally, for any t, as $T \to \infty$, $c_t^*\left(\underline{y}_t\right) \to c_t^{RE}\left(\underline{y}_t\right)$, $\hat{E}\left[H_{t+1}|\underline{y}_t\right] \to E\left[H_{t+1}|\underline{y}_t\right]$, $c_t^*\left(\underline{y}_t\right) \to E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]$.[10] Beliefs become more rational as the stakes become larger.

Second, the agent with optimal expectations may choose not to insure future income given an objectively fair insurance contract. Formally, let the agent face an additional binary decision in period 1: whether or not to exchange all current and future income for $B = E\left[H_1|y_1\right]$. A rational agent would always take this contract, while the agent with optimal expectations may choose not to insure consumption. Interestingly, since beliefs affect whether the agent insures or not, the addition of the possibility of insurance may change what beliefs are optimal.

Optimal expectations are either the beliefs that maximize lifetime well-being conditional

[10] It can be seen that $c_1^*\left(\underline{y}_t\right) \to c_1^{RE}\left(\underline{y}_t\right)$ by repeatedly substituting the Euler equation (13) into the budget constraint to solve for $c_1^*\left(\underline{y}_t\right)$ and noting that $\frac{\psi_{t+\tau}}{\psi_t}R^\tau \to 1$ as $T \to \infty$ and $\frac{\psi_t}{\psi_{t+\tau}}R^{-\tau} < \psi_t$ for $\tau, T \to \infty$. This together with equation (11) imply that $\hat{E}\left[H_2|\underline{y}_1\right] \to E\left[H_2|\underline{y}_1\right]$. Again using the fact that $\frac{\psi_{t+\tau}}{\psi_t}R^\tau \to 1$ as $T \to \infty$, equation (13) implies that for finite t, $c_t^*\left(\underline{y}_t\right) \to c_t^{RE}\left(\underline{y}_t\right)$ so that the first two results also hold for any finite t (not just $t = 1$).

on inducing the agent to accept the insurance, or the beliefs that maximize lifetime well-being conditional on inducing the agent to reject the insurance. The former are the optimal expectations from Proposition 4. These beliefs are optimal for the problem without the constraint, and the agent rejects the insurance because both income streams are perceived as certain and $\hat{E}[H_1|y_1] > E[H_1|y_1] = B$. Lifetime well-being in this case is

$$\sum_{t=1}^{T} \psi_t E\left[u\left(c_t^*\right)|y_1\right]$$

from equation (12). The latter, optimal beliefs conditional on accepting the insurance, are more realistic about the income process. The actual beliefs are irrelevant for lifetime well-being provided that the agent believe that $\hat{E}[H_1|y_1]$ is small enough and/or the process for $\{y\}$ uncertain enough that he accepts the insurance.[11] Lifetime well-being in this case is

$$\sum_{t=1}^{T} \psi_t E\left[u\left(c^{FI}(y_1)\right)|y_1\right]$$

where $c^{FI}(y_1) = \frac{R-1}{R-R^{1-T}} E\left[\sum_{t=1}^{T} R^{1-t} y_t | y_1\right]$.

Risk determines which expectations are optimal. Lifetime well-being decreases in objective income risk when the agent rejects the insurance, while it is invariant to risk if he accepts the insurance. If objective income risk is small, then the cost of distorted beliefs – variable future consumption – is small, and optimal expectations are optimistic. The agent dreams about future income and rejects consumption insurance. If objective income risk is large, optimal expectations are more rational and induce the agent to insure his future income.

Third, at the start of life, the agent facing the problem with the option to insure income may have a lower level of felicity than the agent facing the problem without this option. Informally, we might think of an agent approaching their life blithely optimistic about their future. Given no choice of insurance, this is indeed optimal. However, when faced with the opportunity to insure and in an environment with large amounts of income risk, the agent considers their life more realistically, putting more weight on possible bad states of the world, and chooses insurance. Most interestingly, since $c^*(y_1) > c^{FI}(y_1)$, the agent who chooses to insure is made less happy

[11] While nothing formally requires this, it seems natural to assume that expectations are rational in this case.

today by the option to completely insure income. The agent is happier when the choice set is smaller.[12]

6 Conclusion

This paper introduces a model of utility-serving biases in beliefs. Optimal expectations provides a structural model of non-rational but optimal beliefs. While our applications highlight many of the implications of our theory, many remain to be explored.

First, the specification of possible events seems to be more important in a model with optimal expectations than it is in a model with rational expectations. For example, an optimal expectations equilibrium in a world with only certain outcomes is different from the equilibrium in the same world with an available sunspot or public randomization device. With the randomization device agents can gamble against one another.

Second, agents with optimal expectations can be optimistic about uncertain environments, and therefore can be better off with the later resolution of uncertainty. For instance, you tell someone that they are going to receive gifts on their birthday but not what those gifts are until their birthday.[13] More generally, because more information can change the ability to distort beliefs, agents can be better off not receiving information despite the benefits of better decision making. It is, however, not the case that agents would ever choose that uncertainty be resolved later because agents take their beliefs as given. Bayesian agents never prefer the later resolution of uncertainty.

Third, we conjecture that the agent who faces the same problem again and again, and so faces the possibility of large losses from an incorrect specification of probabilities, will naturally have a better assessment of probabilities. Thus, optimal expectations agents are not easy to turn into "money pumps," although they may exhibit behavior far from that generated by rational expectations in one-shot games.

[12] The agent with the option who accepts the option has greater levels of felicity later in life on average. This is because, lifetime well-being with the option to insure is greater than or equal to lifetime well-being without the option in this model.

[13] A surprise party for an agent raises the possibility in the agent's mind that he might get more surprise parties in the future and he enjoys looking forward to this possibility.

Fourth, and closely related, to what extent do optimal beliefs give an evolutionary advantage or disadvantage relative to rational beliefs? On the one hand, agents with optimal expectations make poorer decisions. On the other hand, agents with optimal expectations may take on more risk, which can lead to an evolutionary advantage.

Finally, optimal expectations has promising applications in strategic environments. In a strategic setting, each agent's beliefs are set taking as given the reaction functions of other agents.

Appendixes

A Optimal expectations when the state space is continuous

In the main text, we describe optimal expectations when the state space is finite and discrete. This appendix defines equilibrium when uncertainty is continuous. Let $\Pi\left(s_t|\underline{s}_{t-1}\right)$ denote the conditional cumulative probability distribution function of the vector $s_t \in S$ and $\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right)$ the subjective version. When the state space is continuous, equation (4) becomes

$$V^*\left(x_t;\underline{s}_t,\left\{\hat{\Pi}\right\}\right)=\max_{c_t}\ u\left(x_t,c_t\right)+\beta\int V^*\left(g\left(x_t,c_t,s_{t+1}\right);s_{t+1},\underline{s}_t,\left\{\hat{\Pi}\right\}\right)d\hat{\Pi}\left(s_{t+1}\right).$$

Assumption 1 is replaced by

Assumption 1´ (Restrictions on probabilities, continuous state space)

(i) $\int_{s_t\in S} d\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right)=1$

(ii) $d\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right)\geq 0$

(iii) $\hat{\Pi}\left(\underline{s}'_t\right)=\hat{\Pi}\left(s'_t|\underline{s}'_{t-1}\right)\hat{\Pi}\left(s'_{t-1}|\underline{s}'_{t-2}\right)\cdots\hat{\Pi}\left(s'_1\right)$

(iv) $d\hat{\Pi}\left(s_t|\underline{s}_{t-1}\right)=0$ if $d\Pi\left(s_t|\underline{s}_{t-1}\right)=0$.

Finally, optimal expectations are determined by choosing continuous probability functions to maximize the functional objective

$$\max_{\left\{\hat{\Pi}\left(s_\tau|\underline{s}_{\tau-1}\right)\right\}}\ E\left[\sum_{t=1}^{T}\beta^{t-1}\left(M\left(\underline{x}^*\left(\underline{s}_{t-1},\left\{\hat{\Pi}\right\}\right),\underline{c}^*\left(\underline{s}_{t-1},\left\{\hat{\Pi}\right\}\right)\right)+V^*\left(x^*\left(\underline{s}_t,\left\{\hat{\Pi}\right\}\right);\underline{s}_t,\left\{\hat{\Pi}\right\}\right)\right)\right].$$

B Proofs of Propositions

B.1 Proof of Proposition 1

Proof: (i) We prove the case for $w^{OE}>0$; the case for $w^{OE}<0$ is analogous. If $w^{OE}>w^{RE}$, then

$$\begin{aligned} u'\left(R+w^{OE}Z_s\right) &\geq u'\left(R+w^{RE}Z_s\right)\ \text{for } s\ni Z_s\leq 0 \\ u'\left(R+w^{OE}Z_s\right) &< u'\left(R+w^{RE}Z_s\right)\ \text{for } s\ni Z_s>0 \end{aligned} \tag{B.1}$$

When the asset pays off poorly, marginal utility is higher for the agent with the higher share invested in the risky asset. The agent with rational expectations has first order condition

$$\sum_{s\ni Z_s\leq 0}\pi_s u'\left(R+w^{RE}Z_s\right)Z_s+\sum_{s\ni Z_s>0}\pi_s u'\left(R+w^{RE}Z_s\right)Z_s=0$$

which combined with equation (B.1) implies

$$\sum_{s \ni Z_s \leq 0} \pi_s u' \left(R + w^{OE} Z_s\right) Z_s + \sum_{s \ni Z_s > 0} \pi_s u' \left(R + w^{OE} Z_s\right) Z_s < 0$$

Subtracting this from the first-order condition of the agent with optimal expectations yields the desired inequality

$$\sum_{s=1}^{S} \left(\hat{\pi}_s^{OE} - \pi_s\right) u' \left(R + w^{OE} Z_s\right) Z_s > 0 \tag{B.2}$$

Thus if we can show that $w^{OE} > 0$ implies $w^{OE} > w^{RE}$ we complete the proof of (i). This follows from the second point of the proposition, which we now prove.

(ii) We first treat the case $w^{RE} > 0$, the case $w^{RE} < 0$ is analogous and we treat $w^{RE} = 0$ subsequently. We first show that an agent invests less in the risky asset as the probability of the worst state is increased ($\frac{dw^*}{d\hat{\pi}_1} < 0$). Examine the agent's first-order condition for w^* and consider a small increase in $\hat{\pi}_{s'}$ at the expense of $\hat{\pi}_S$, so for all $s' \leq S$:

$$\begin{aligned} 0 &= \left(u'\left(R + w^* Z_{s'}\right) Z_{s'} - u'\left(R + w^* Z_S\right) Z_S\right) d\hat{\pi}_{s'} + \sum_{s=1}^{S} \hat{\pi}_s u''\left(R + w^* Z_s\right) Z_s^2 dw^* = 0 \\ \frac{dw^*}{d\hat{\pi}_{s'}} &= \frac{u'\left(R + w^* Z_S\right) Z_S - u'\left(R + w^* Z_{s'}\right) Z_{s'}}{\sum_{s=1}^{S} \hat{\pi}_s u''\left(R + w^* Z_s\right) Z_s^2} \end{aligned}$$

Note that this equals zero for $s' = S$ and that the denominator is always negative. When the numerator is positive, as for example when $Z_{s'} < 0$, then when the probability of state s' increases at the expense of the best state, the optimal portfolio share in the risky asset decreases. In particular,

$$\frac{dw^*}{d\hat{\pi}_1} < 0$$

When subjective probability moves from the best to the worst state, the optimal portfolio share declines (short position increases).

Suppose for purposes of contradiction, that $0 < w^{OE} \leq w^{RE}$. Since $w^{OE} \leq w^{RE}$, we can follow the same steps as in (i) to show that

$$\sum_{s=1}^{S} \left(\hat{\pi}_s^{OE} - \pi_s\right) u' \left(R + w^{OE} Z_s\right) Z_s \leq 0 \tag{B.3}$$

We also have predictions about the sign of the left hand side of equation (B.3) from equation (7) – the optimal choice of $\hat{\pi}_1^{OE}$, which implies

$$\begin{aligned} \text{sign}\left[\beta\left(u_S - u_1\right)\right] &= \text{sign}\left[\frac{dw^*}{d\hat{\pi}_1}\sum_{s=1}^{S}\pi_s u'\left(R + w^{OE}Z_s\right)Z_s\right] \\ &= \text{sign}\left[-\sum_{s=1}^{S}\pi_s u'\left(R + w^{OE}Z_s\right)Z_s\right] \\ &= \text{sign}\left[\sum_{s=1}^{S}\left(\hat{\pi}_s^{OE} - \pi_s\right)u'\left(R + w^{OE}Z_s\right)Z_s\right] \end{aligned}$$

where the second equality uses $\frac{dw^*}{d\hat{\pi}_1} < 0$ and the third makes use of the agent's first order condition (equation (6)). When the agent is investing in the risky asset $u_S - u_1 > 0$, and when shorting, $u_S - u_1 < 0$ so that

$$\text{sign}\left[w^{OE}\right] = \text{sign}\left[\sum_{s=1}^{S}\left(\hat{\pi}_s^{OE} - \pi_s\right)u'\left(R + w^{OE}Z_s\right)Z_s\right] \tag{B.4}$$

which with (B.3) implies $w^{OE} \leq 0$. Thus we have a contradiction and we know for $w^{RE} > 0$, $w^{OE} > w^{RE}$ or $w^{OE} \leq 0$. The final step is to rule out $w^{OE} = 0$. If $w^{OE} = 0$ then $\beta\left(u_S - u_1\right) = 0$ so that by the first-order condition for beliefs

$$0 = \sum_{s=1}^{S}\pi_s u'\left(R + w^{OE}Z_s\right)Z_s. \tag{B.5}$$

But since the first-order condition of the rational agent is equation (B.5) with w^{RE} instead of w^{OE} and that this yields a unique $w^{RE} > 0$ we know that $w^{OE} = 0$ cannot solve equation (B.5).

Finally, consider the case $w^{RE} = 0$. This occurs if and only if $E[Z] = 0$. Suppose that the optimal expectations equilibrium portfolio is $w^{\Diamond} = 0$, and denote optimal beliefs $\hat{\pi}_s^{\Diamond}$, then $\hat{E}[Z] = 0$. In fact, such beliefs solve the first-order conditions for optimal beliefs because when $w^{\Diamond} = 0$, $u_S - u_{s'} = 0$ and $\sum_{s=1}^{S}\hat{\pi}_s^{\Diamond}u'(R)Z_s = u'(R)\hat{E}[Z_s] = 0$. But first order conditions are not sufficient here; the second-order condition is violated. Consider total lifetime well-being as a function of w^* and $\hat{\pi}_1$, $W^*(w^*(\hat{\pi}_1), \hat{\pi}_1)$ and take the deviation from $\left\{\hat{\pi}^{\Diamond}\right\}$ that moves probability from the best state to the worst, $\hat{\pi}_1 = \hat{\pi}_1^{\Diamond} + d\hat{\pi}$ and $\hat{\pi}_S = \hat{\pi}_S^{\Diamond} - d\hat{\pi}$. The change in lifetime well-being for this $d\hat{\pi}$ (using $w^*\left(\hat{\pi}^{\Diamond}\right) = 0$) is

$$d\mathcal{W}^*(w^*\left(\hat{\pi}_1^{\Diamond}\right), \hat{\pi}_1^{\Diamond}) = \mathcal{W}_1^*\frac{dw^*}{d\hat{\pi}}d\hat{\pi} + \mathcal{W}_2^*d\hat{\pi} + \mathcal{W}_{12}^*\frac{dw^*}{d\hat{\pi}}\left(d\hat{\pi}\right)^2 + \mathcal{W}_1^*\frac{d^2w^*}{d\hat{\pi}^2}\left(d\hat{\pi}\right)^2 + \mathcal{W}_{22}^*\left(d\hat{\pi}\right)^2.$$

where $\mathcal{W}_i^*$ represents a partial derivative. The first-order terms are zero (since the first-order condition is satisfied), $\mathcal{W}^*{}_{22} = 0$ since the problem is linear is probabilities, and $\mathcal{W}_1^* = \beta \sum_{s=1}^{S} \hat{\pi}_s^{\Diamond} u'(R) Z_s + \beta \sum_{s=1}^{S} \pi_s u'(R) Z_s = 0$ since $\hat{E}[Z] = 0$ and $E[Z] = 0$. Thus we have

$$
\begin{aligned}
d\mathcal{W}^*(w^*\left(\hat{\pi}_1^{\Diamond}\right), \hat{\pi}_1^{\Diamond}) &= -\beta u'(R)(Z_S - Z_1)\frac{dw^*}{d\hat{\pi}}(d\hat{\pi})^2 \\
&= -\frac{\beta\left(u'(R)(Z_S - Z_1)\right)^2}{u''(R)\sum_{s=1}^{S}\hat{\pi}_s^{\Diamond} Z_s^2}(d\hat{\pi})^2 > 0
\end{aligned}
$$

which implies that the change in lifetime well-being is positive so that $w^{OE} \neq w^{\Diamond} = 0$. QED.

B.2 Proof of Proposition 2

We show that as $\pi_1 \to 1$, lifetime well-being investing in the asset is higher than when shorting the asset. We do this by constructing a lower bound for lifetime well-being when investing in the asset ($\underline{\mathcal{W}}^+(\pi_1)$) and an upper bound when shorting the asset ($\overline{\mathcal{W}}^-(\pi_1)$) and showing that $\lim_{\pi_1 \to \infty} \underline{\mathcal{W}}^+(\pi_1) > \lim_{\pi_1 \to \infty} \overline{\mathcal{W}}^-(\pi_1)$. Define lifetime well-being as a function of subjective and objective beliefs, given optimal agent behavior, as

$$
\mathcal{W}(\hat{\pi}_1; \pi_1) := (\pi_1 + \hat{\pi}_1)\, u(R + w^* Z_1) + (2 - \pi_1 - \hat{\pi}_1)\, u(R + w^* Z_2)
$$

where $Z_s = Z_s(\pi_1)$, and $w^* = w^*(\hat{\pi}_1; Z_1(\pi_1), Z_2(\pi_1))$.

Step 1: $\lim_{\pi_1 \to 1} W(\cdot) = \infty$ for $w > 0$.

Take an arbitrary $\hat{\pi}_1'$ such that $0 < \hat{\pi}_1' < \pi_1$ and denote the associated portfolio choice w'. The quantity

$$
\underbrace{(\hat{\pi}_1' - \pi_1)}_{-}\underbrace{u'\left(R + w' Z_1\right)}_{+}\underbrace{Z_1}_{-} - \underbrace{(\hat{\pi}_1' - \pi_1)}_{-}\underbrace{u'\left(R + w' Z_1\right)}_{+}\underbrace{Z_2}_{+}
$$

is positive, which implies $w' > 0$ by Proposition 1. Since $\hat{\pi}'$ may be suboptimal, $W\left(\hat{\pi}_1'; \pi_1\right)$ is a lower bound for the lifetime well-being of the agent conditional on $w > 0$, $W\left(\hat{\pi}_1'; \pi_1\right) \leq W\left(\hat{\pi}_1(\pi_1), \pi_1\right)$. Define $\underline{\mathcal{W}}^+(\pi_1) := W\left(\hat{\pi}_1'; \pi_1\right)$. Taking the limit as skewness goes to infinity,

$$
\begin{aligned}
\lim_{\pi_1 \to 1} \underline{\mathcal{W}}^+(\pi_1) &= \left(1 + \hat{\pi}_1'\right)\lim_{\pi_1 \to 1} u\left(R + w' Z_1\right) + \left(1 - \hat{\pi}_1'\right)\lim_{\pi_1 \to 1} u\left(R + w' Z_2\right) \\
&= \left(1 + \hat{\pi}_1'\right) u\left(R + w' \mu_Z\right) + \left(1 - \hat{\pi}_1'\right)\lim_{\pi_1 \to 1} u\left(R + w' Z_2\right) \\
&= \infty
\end{aligned}
$$

since $Z_1 \to \mu_Z$, $Z_2 \to \infty$ and $\lim_{c\to\infty} u(c) = \infty$.

Step 2: $\lim_{\pi_1 \to 1} W(\cdot) < \infty$ for $w < 0$.

Define an upper bound for lifetime well-being by choosing the portfolio and subjective beliefs subject only to the conditions that the agent short the asset, that the agent is pessimistic about the payout, and that the portfolio be feasible:

$$\begin{aligned}
\overline{\mathcal{W}}^{-}(\pi_1) \quad &: \quad = \sup_{w,\hat{\pi}_1} (\pi_1 + \hat{\pi}_1)\, u(R + wZ_1) + (2 - \pi_1 - \hat{\pi}_1)\, u(R + wZ_2) \\
\text{s.t.} \quad & \quad w \le 0 \\
& \quad \hat{\pi}_1 > \pi_1 \\
& \quad R + wZ_2 \ge 0
\end{aligned}$$

$\overline{\mathcal{W}}^{-}(\pi_1)$ is an upper bound since we do not restrict w to be the optimal agent's choice given $\hat{\pi}_1$. It is easy to see that the optimal $\hat{\pi}_1 = 1$. Hence,

$$\begin{aligned}
\lim_{\pi_1 \to 1} \overline{\mathcal{W}}^{-}(\pi_1) &= \lim_{\pi_1 \to 1} \sup_{-\frac{R}{Z_2} \le w \le 0} (1 + \pi_1)\, u(R + wZ_1) + (1 - \pi_1)\, u(R + wZ_2) \\
&= \lim_{\pi_1 \to 1} (1 + \pi_1)\, u\left(R + \left(-\frac{R}{Z_2}\right) Z_1\right) + (1 - \pi_1)\, u(R) \\
&= 2u(R)
\end{aligned}$$

where the second step follows from the maximization, and the third from the fact that $\lim_{\pi_1 \to 1} \frac{Z_1}{Z_2} = 0$.

Both parts of the proposition follow from $\lim_{\pi_1 \to 1} \underline{\mathcal{W}}^{+}(\pi_1) > \lim_{\pi_1 \to 1} \overline{\mathcal{W}}^{-}(\pi_1)$ and Proposition 1.

B.3 Proof of Proposition 3

(i) For rational expectations, each agent's first order condition is

$$\sum_s \pi_s \left(\frac{1 + \varepsilon_s}{P} - 1\right) u'\left(\left(X_b^i + PX_e^i\right)\left(1 + w^{RE,i}\left(\frac{1 + \varepsilon_s}{P} - 1\right)\right)\right) = 0.$$

We guess that the market-clearing price is

$$P = 1 + \sum_s \pi_s \varepsilon_s.$$

Substituting into the above first order condition shows after some algebra

$$u'\left(c_1^{RE,i}\right) - u'\left(c_2^{RE,i}\right) = 0$$

which implies that consumption is perfectly smoothed across states since $u'(\cdot)$ is monotonic. Writing $c_1^{RE,i} = c_2^{RE,i}$ out yields

$$\begin{aligned}
\left(X_b^i + PX_e^i\right)\left(1 + w^{RE,i}\left(\frac{1+\varepsilon_1}{P} - 1\right)\right) &= \left(X_b^i + PX_e^i\right)\left(1 + w^{RE,i}\left(\frac{1+\varepsilon_2}{P} - 1\right)\right) \\
w^{RE,i}\left(\frac{1+\varepsilon_1}{1+\sum_s \pi_s \varepsilon_s} - 1\right) &= w^{RE,i}\left(\frac{1+\varepsilon_2}{1+\sum_s \pi_s \varepsilon_s} - 1\right).
\end{aligned}$$

This implies $w^{RE,i} = 0$ unless the return on the stochastic asset is the same across states, $\frac{1+\varepsilon_1}{1+\sum_s \pi_s\varepsilon_s} = \frac{1+\varepsilon_2}{1+\sum_s \pi_s\varepsilon_s}$. It is straightforward that $x_e^{RE,i} = 0$ and that these allocations satisfy the aggregate resource constraint.

(ii) Proof by contradiction that $w^{OE,i} \neq 0$ for both agents, using equations 8c. Suppose $w^{OE,i} = 0$, then consumption is perfectly smoothed across states, $c_s^i = c$. Some algebra shows that this allocation can satisfy the first order conditions and the resource constraint for rational beliefs and equilibrium price. But these conditions are not sufficient and this allocation is not an optimal allocation. We show that each agent can change his beliefs and allocations slightly within the feasible set and increase his lifetime well-being. Consider the deviation from objective beliefs, $\hat{\pi}_s^{\diamond,i} = \pi_s + (-1)^{i+s}\, d\hat{\pi}$. In this case agents respond by increasing (decreasing) their consumption in the state they perceive as more (less) likely by dc so as to satisfy their first-order condition.

$$0 = \sum_s \left[(-1)^{i+s} u'(c)(1+\varepsilon_s - P)\, d\hat{\pi} + \pi_s u''(c) \frac{dc_s^{i*}}{dc_1^{i*}} \frac{dc_1^{i*}}{d\hat{\pi}} d\hat{\pi}\,(1+\varepsilon_s - P)\right]$$

The symmetry of the deviation maintains P at the rational expectations price, so that it is clear that the aggregate resource constraint is met. By the budget constraint, for either agent

$$\begin{aligned}
\frac{dc_1}{dc_2} &= \frac{1+\varepsilon_1 - P}{1+\varepsilon_2 - P} \\
&= \frac{\varepsilon_1 - \pi_1\varepsilon_1 - \pi_2\varepsilon_2}{\varepsilon_2 - \pi_1\varepsilon_1 - \pi_2\varepsilon_2} \\
&= -\frac{\pi_2}{\pi_1}
\end{aligned}$$

Thus, after some algebra

$$\frac{dc_1^{i*}}{d\hat{\pi}} = (-1)^i \frac{u'(c)}{u''(c)\,\pi_1}.$$

Let lifetime well-being be $W^{*i}\left(c_1^{i*}(\hat{\pi}_1), \hat{\pi}_1^i\right)$ in which we are treating c_2^{i*} as a function of c_1^{i*}. The change in lifetime well-being for agent i and the change $d\hat{\pi}$ has zero first-order effects, as in the partial-equilibrium case. The second order terms are

$$\mathcal{W}_{12}^{*i} \frac{dc_1^{i*}}{d\hat{\pi}} (d\hat{\pi})^2 + \mathcal{W}_1^{*i} \left(\frac{dc_1^{i*}}{d\hat{\pi}} \right)^2 (d\hat{\pi})^2 + \mathcal{W}_{22}^{*i} (d\hat{\pi})^2 .$$

$W_{22}^* = 0$ since W^* is linear in probabilities.

$$\mathcal{W}_1^{*i} = \sum_s \left[\beta \hat{\pi}_s^i u' \left(c_s^i \right) \frac{dc_s^{i*}}{dc_1^{i*}} + \beta \pi_s u' \left(c_s^i \right) \frac{dc_s^{i*}}{dc_1^{i*}} \right] = 0$$

where the last equality comes from evaluating at $c_s^i = c$ and substituting $\frac{dc_1}{dc_2}$. Thus the total welfare gain simplifies to

$$\begin{aligned} \mathcal{W}_{12}^{*i} \frac{dc_1^{i*}}{d\hat{\pi}} (d\hat{\pi})^2 &= 2\beta (-1)^{i+1} u'(c) \left(1 + \frac{\pi_1}{\pi_2} \right) (-1)^i \frac{u'(c)}{u''(c)\, \pi_1} (d\hat{\pi})^2 \\ &= 2\beta \frac{(u'(c))^2 \left(\frac{1}{\pi_1} + \frac{1}{\pi_2} \right)}{-u''(c)} (d\hat{\pi})^2 > 0. \end{aligned}$$

Intuitively, in this alternative allocation each agent suffers a second-order welfare loss in the second period for standard arguments about perfect consumption smoothing. However, the overall welfare gain is positive because the agent can invest in the state that he believes more likely and anticipate a utility gain. This yields a contradiction so that we have $w^1 \neq w^2$. Since at least one agent has $w^{OE} \neq 0$, both agents have $w^{OE} \neq 0$ by the aggregate resource constraint and noting that each agent has a positive endowment. So agents are gambling.

(iii) We proceed in 5 steps.

Step 1 $\mu^i \neq 0$:

Suppose $\mu^i = 0$, then $\beta u_s \left(c_s^{OE,i} \right) = \lambda^i$ for both s by the first order condition $\left(\hat{\pi}_s^{OE,i} \right)$ (equation 8b). Since λ^i does not depend on s, this implies that the agent would perfectly smooth consumption across states, $u \left(c_1^{OE,i} \right) = u \left(c_2^{OE,i} \right)$. This in turn, implies $w^{OE,i} = 0$, as shown above, which by the aggregate resource constraint implies the other agent's portfolio share in the risky asset is also zero. A contradiction arises and hence, we must have $\mu^i \neq 0$ at the optimum.

Step 2: there are then two cases, case (a) $c_1^{OE,1} > c_2^{OE,1}$ and $c_2^{OE,2} > c_1^{OE,2}$ and case (b) $c_2^{OE,1} > c_1^{OE,1}$ and $c_1^{OE,2} > c_2^{OE,2}$.

The μ^i equations imply that $(1 + \varepsilon_2) > P > (1 + \varepsilon_1) > 0$ since $\hat{\pi} \geq 0$ and $u' > 0$. The aggregate resource constraint implies that (a) $w^{OE,2} > 0 > w^{OE,1}$ or (b) $w^{OE,1} > 0 > w^{OE,2}$. These in turn imply by the consumption definition in (a) $c_1^{OE,1} = \left(X_b^1 + PX_e^1\right)\left(1 + w^{OE,1}\left(\frac{1+\varepsilon_1}{P} - 1\right)\right) > c_2^{OE,1} = \left(X_b^1 + PX_e^1\right)\left(1 + w^{OE,1}\left(\frac{1+\varepsilon_2}{P} - 1\right)\right)$ and $c_2^{OE,2} > c_1^{OE,2}$; case (b) $c_2^{OE,1} > c_1^{OE,1}$ and $c_1^{OE,2} > c_2^{OE,2}$.

Step 3: in case (a) $\mu^1 > 0 > \mu^2$ and in case (b) $\mu^2 > 0 > \mu^1$.

The next step provides more information about μ^i. From equations $\hat{\pi}_s^i$

$$\begin{aligned} 0 &= \beta u\left(c_1^{OE,i}\right) - \lambda^i - \mu^i\left[u'\left(c_1^{OE,i}\right)\left(\frac{1+\varepsilon_1}{P} - 1\right)P\right] \\ 0 &= \beta u\left(c_2^{OE,i}\right) - \lambda^i - \mu^i\left[u'\left(c_2^{OE,i}\right)\left(\frac{1+\varepsilon_2}{P} - 1\right)P\right] \end{aligned}$$

subtract the equations for the same agent

$$\beta u\left(c_1^{OE,i}\right) - \beta u\left(c_2^{OE,i}\right) = \mu^i P\left[u'\left(c_1^{OE,i}\right)\left(\frac{1+\varepsilon_1}{P} - 1\right) - u'\left(c_2^{OE,i}\right)\left(\frac{1+\varepsilon_2}{P} - 1\right)\right]$$

Case (a) $w^{OE,2} > 0 > w^{OE,1}$, $c_1^{OE,1} > c_2^{OE,1}$ and $c_2^{OE,2} > c_1^{OE,2}$ the left hand side is positive for $i = 1$, negative for $i = 2$, so

$$\begin{aligned} \mu^1\left[u'\left(c_1^{OE,1}\right)(1+\varepsilon_1 - P) - u'\left(c_2^{OE,1}\right)(1+\varepsilon_2 - P)\right] &> 0 \\ \mu^2\left[u'\left(c_1^{OE,2}\right)(1+\varepsilon_1 - P) - u'\left(c_2^{OE,2}\right)(1+\varepsilon_2 - P)\right] &< 0 \end{aligned}$$

Now the two first order conditions for w^i :

$$0 = \sum_s\left[\left(\beta\hat{\pi}_s^{OE,i} + \beta\pi_s\right)u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P} - 1\right)\right] + \mu^i\,[+]$$

where $[+]$ is some positive number.

For $i = 1$, write out and substitute π_2 and $\hat{\pi}_2$ using the fact the probabilities sum to one, and multiply through by μ^i, giving

$$\begin{aligned} 0 &= \left(\hat{\pi}_1^{OE,i} + \pi_1\right)u'\left(c_1^{OE,i}\right)\left(\tfrac{1+\varepsilon_1}{P} - 1\right) + 2u'\left(c_2^{OE,i}\right)\left(\tfrac{1+\varepsilon_2}{P} - 1\right) \\ &\quad - \left(\hat{\pi}_1^{OE,i} + \pi_1\right)u'\left(c_2^{OE,i}\right)\left(\tfrac{1+\varepsilon_2}{P} - 1\right) + \mu^i\,[+] \\ &= 2\mu^i u'\left(c_2^{OE,i}\right)(1+\varepsilon_2 - P) \\ &\quad + \left(\hat{\pi}_1^{OE,i} + \pi_1\right)\mu^i\left\{u'\left(c_1^{OE,i}\right)(1+\varepsilon_1 - P) - u'\left(c_2^{OE,i}\right)(1+\varepsilon_2 - P)\right\} + \left(\mu^i\right)^2[+]\,. \end{aligned}$$

The inequality above implies the term $\mu^1\{\} > 0$ so that we have

$$0 = [+] + \mu^i\,[+] + [+]$$

so that $\mu^1 < 0$.

Similarly, for $i = 2$, we get

$$\begin{aligned}
0 &= 2u'\left(c_1^{OE,2}\right)\left(\tfrac{1+\varepsilon_1}{P}-1\right) - \left(\hat{\pi}_2^{OE,2}+\pi_2\right)u'\left(c_1^{OE,2}\right)\left(\tfrac{1+\varepsilon_1}{P}-1\right) \\
&\quad + \left(\hat{\pi}_2^{OE,2}+\pi_2\right)u'\left(c_2^{OE,2}\right)\left(\tfrac{1+\varepsilon_2}{P}-1\right) + \mu^i\left[+\right] \\
&= 2\mu^i u'\left(c_1^{OE,2}\right)\left(1+\varepsilon_1-P\right) \\
&\quad - \left(\hat{\pi}_2^{OE,2}+\pi_2\right)\mu^i\left\{u'\left(c_1^{OE,2}\right)\left(1+\varepsilon_1-P\right)+u'\left(c_2^{OE,2}\right)\left(1+\varepsilon_2-P\right)\right\} + \left(\mu^i\right)^2\left[+\right]
\end{aligned}$$

The inequality above implies the term $\mu^i\{\} < 0$ so that we have

$$0 = \mu^2\left[-\right] - \left[-\right] + \left[+\right]$$

so that $\mu^2 > 0$.

Step 4: Optimism about high consumption state

Turning to probabilities, from the first equation, $0 = \sum_s\left(\beta\hat{\pi}_s^{OE,i}+\beta\pi_s\right)u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P}-1\right) - \mu^i\left[-\right]$ since $u'' < 0$ so that

$$\text{sign}\left\{\sum_s\left(\beta\hat{\pi}_s^{OE,i}+\beta\pi_s\right)u'\left(c_s^{OE,i}\right)\left(\frac{1+\varepsilon_s}{P}-1\right)\right\} = \text{sign}\left\{-\mu^i\right\}$$

For $i = 1$, $c_1^{OE,1} > c_2^{OE,1}$ so $u'\left(c_1^{OE,1}\right) < u'\left(c_2^{OE,1}\right)$

$$\left(\beta\hat{\pi}_1^{OE,1}+\beta\pi_1\right)u'\left(c_1^{OE,1}\right)\left(\frac{1+\varepsilon_1}{P}-1\right) + \left(\beta\hat{\pi}_2^{OE,1}+\beta\pi_2\right)u'\left(c_2^{OE,1}\right)\left(\frac{1+\varepsilon_2}{P}-1\right) > 0$$

From the agent's optimization problem, $\sum_s\hat{\pi}_s^{OE,i}u'\left(c_s^{OE,i}\right)\left((1+\varepsilon_s)-P\right) = 0$. This implies

$$\pi_1 u'\left(c_1^{OE,1}\right)\left(\frac{1+\varepsilon_1}{P}-1\right) + \pi_2 u'\left(c_2^{OE,1}\right)\left(\frac{1+\varepsilon_2}{P}-1\right) > 0$$

and again

$$\left(\pi_1-\hat{\pi}_1^{OE,1}\right)u'\left(c_1^{OE,1}\right)\left(\frac{1+\varepsilon_1}{P}-1\right) + \left(\pi_2-\hat{\pi}_2^{OE,1}\right)u'\left(c_2^{OE,1}\right)\left(\frac{1+\varepsilon_2}{P}-1\right) > 0$$

and using the fact that probabilities sum to unity

$$\begin{aligned}
\left(\pi_1-\hat{\pi}_1^{OE,1}\right)u'\left(c_1^{OE,1}\right)\left(\frac{1+\varepsilon_1}{P}-1\right) + \left(1-\pi_1-1+\hat{\pi}_1^{OE,1}\right)u'\left(c_2^{OE,1}\right)\left(\frac{1+\varepsilon_2}{P}-1\right) &> 0 \\
\left(\pi_1-\hat{\pi}_1^{OE,1}\right)\left\{u'\left(c_1^{OE,1}\right)\left(\frac{1+\varepsilon_1}{P}-1\right) - u'\left(c_2^{OE,1}\right)\left(\frac{1+\varepsilon_2}{P}-1\right)\right\} &> 0
\end{aligned}$$

Finally, we know $\mu^1\left[u'\left(c_1^{OE,1}\right)(1+\varepsilon_1-P)-u'\left(c_2^{OE,1}\right)(1+\varepsilon_2-P)\right]>0$ and $u'\left(c_1^{OE,1}\right)(1+\varepsilon_1-P)-u'\left(c_2^{OE,1}\right)(1+\varepsilon_2-P)<0$ so we have $\left(\pi_1-\hat{\pi}_1^{OE,1}\right)<0$ or $\hat{\pi}_1^{OE,1}>\pi_1$. A symmetric argument shows $\hat{\pi}_1^{OE,2}<\pi_1$ and the probabilities of the second state follows.

B.4 Proof of Proposition 4

Part (ii): Subtract $E\left[c_{t+1}^{OE}|\underline{y}_t\right]$ from each side of equation (13) with $\tau=1$

$$\begin{aligned}
c_t^{OE}-E\left[c_{t+1}^{OE}|\underline{y}_t\right] &= \frac{a}{b}-\frac{\psi_{t+1}}{\psi_t}R\left(\frac{a}{b}-E\left[c_{t+1}^{OE}|\underline{y}_t\right]\right)-E\left[c_{t+1}^{OE}|\underline{y}_t\right]\\
&= \left(1-\frac{\psi_{t+1}}{\psi_t}R\right)\left(\frac{a}{b}-E\left[c_{t+1}^{OE}|\underline{y}_t\right]\right)
\end{aligned}$$

Since the support of the income process does not admit a plan such that $E\left[c_{t+\tau}^{OE}|\underline{y}_t\right]>\frac{a}{b}$ the second term is positive. The following demonstrates that the first term is positive.

$$\begin{aligned}
\frac{\psi_{t+1}}{\psi_t}R &= \frac{\beta^t}{\beta^{t-1}}R\frac{1+\sum_{\tau=1}^{T-t-1}\left(\beta^\tau+\delta^\tau\right)}{1+\sum_{\tau=1}^{T-t}\left(\beta^\tau+\delta^\tau\right)}\\
&< \beta R\frac{1+\sum_{\tau=1}^{T-t-1}\left(\beta^\tau+\delta^\tau\right)}{1+\sum_{\tau=1}^{T-t-1}\left(\beta^\tau+\delta^\tau\right)+\left(\beta^{T-t}+\delta^{T-t}\right)}\\
&< 1
\end{aligned}$$

therefore

$$c_t^{OE}-E\left[c_{t+1}^{OE}|\underline{y}_t\right]>0$$

and we have the result.

Part (iii): From the agent's consumption Euler equation

$$\hat{E}\left[c_{t+1}^*\left(A_{t+1},\underline{y}_{t+1}\right)|\underline{y}_t\right]=c_t^*\left(A_t,\underline{y}_t\right)>E\left[c_{t+1}^*\left(\underline{y}_{t+1}\right)|\underline{y}_t\right]$$

where the inequality follows from part (ii).

Part (i): the consumption rule at $t+1$ is

$$c_{t+1}^{OE}=\tfrac{1-R^{-1}}{1-R^{-(T-t)}}\left(A_{t+1}+\hat{E}\left[H_{t+1}|\underline{y}_{t+1}\right]\right)$$

where $H_{t+1}=\sum_{\tau=0}^{T-t-1}R^{-\tau}y_{t+\tau+1}$. From part (iii) we have

$$\begin{aligned}
\hat{E}\left[c_{t+1}^{OE}|\underline{y}_t\right] &> E\left[c_{t+1}^{OE}|\underline{y}_t\right]\\
\hat{E}\left[\tfrac{1-R^{-1}}{1-R^{-(T-t)}}\left(A_{t+1}+\hat{E}\left[H_{t+1}|\underline{y}_{t+1}\right]\right)|\underline{y}_t\right] &> E\left[\tfrac{1-R^{-1}}{1-R^{-(T-t)}}\left(A_{t+1}+\hat{E}\left[H_{t+1}|\underline{y}_{t+1}\right]\right)|\underline{y}_t\right]
\end{aligned}$$

and by the law of iterated expectations

$$\hat{E}\left[H_{t+1}|\underline{y}_t\right] > E\left[\hat{E}\left[H_{t+1}|\underline{y}_{t+1}\right]|\underline{y}_t\right]$$

QED.

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