New Results in 2-D Systems Theory, Part II: 2-D State-Space Models—Realization and the Notions of Controllability, Observability, and Minimality

SUN-YUAN KUNG, STUDENT MEMBER, IEEE, BERNARD C. LÉVY, STUDENT MEMBER, IEEE, MARTIN MORF, MEMBER, IEEE, AND THOMAS KAILATH, FELLOW, IEEE

Abstract—In this part, a comparison between the different state-space models is presented. We discuss proper definitions of state, controllability and observability and their relations to minimality of 2-D systems. We also present new circuit realizations and 2-D digital filter hardware implementation of 2-D transfer functions.

I. Introduction

During recent years, several authors (Attasi, Fornasini–Marchesini, Givone–Roesser) have proposed different state-space models for 2-D systems and have suggested some extensions of the usual 1-D notions of controllability, observability, and minimality to the 2-D case. However, these results are not quite satisfactory; they either lack motivation for the state-space models introduced or the notion of state-space is improperly defined.

In this paper, we try to provide some answers to these questions from a practical as well as algebraic standpoint. In Section II, we start with a comparison of all the current models based on a practical (circuit-oriented) point of view and on a proper definition of state. It is shown that the model of Givone–Roesser is the most satisfactory in that respect: it is also the most general since the Attasi and Fornasini–Marchesini models can be imbedded in the Givone–Roesser model.

Also, from the circuit point of view, we present in Section III an implementation of 2-D transfer functions using two types of dynamic elements, i.e., horizontal delay elements z^{-1} and vertical delay elements ω^{-1}. The hardware implementation of 2-D digital filters for imaging systems is also discussed in Section III.

In Section II we point out that a major difference between 1-D and 2-D systems is that in the 2-D case a *global* state (which preserves all past information) and a *local* state (which gives us the size of the recursions of the 2-D filter) can be introduced. In Section IV we discuss the corresponding properties of global and local controllability (observability) and show that these notions are not satisfactory from the point of view of minimality.

However, a more algebraic approach based on eigencurves and eigencones enables us in Section V to introduce the concept of *modal* controllability (observability). We show that a system is minimal if and only if (iff) it is modally observable and controllable.

Manuscript received December 5, 1976. This work was supported by the Advanced Research Projects Agency through the facilities at the Stanford Artificial Intelligence Laboratory, in part by the National Science Foundation under Contract NSF-ENG75-18952, in part by the Air Force Office of Scientific Research, Air Force Systems Command, under Contract AF 44-620-69-C-0101, and in part by the Joint Services Electronics Program under Contract N-00014-67-A-0112-0044.

The authors are with the Information Systems Laboratory, Stanford University, Stanford, CA 94305.

It is then natural to ask whether minimal 2-D realizations exist. If we are given an irreducible transfer function of order (n, m), a state-space realization is minimal iff it is of size $n + m$. The existence of such $(n + m)$ (real or complex) realizations is discussed in Section VI.

We show how Roesser's model arises from the algebraic point of view of Nerode equivalence. In conclusion, it appears that the results obtained by the algebraic and the practical approaches are quite compatible.

II. State-Space Models for 2-D Systems

During recent years, several authors: Attasi [1], [2], Fornasini–Marchesini [3], [4] and Givone–Roesser [5] have proposed different state-space models for 2-D systems. This apparent divergence of the results so far arises from the fact that the previous authors have investigated the problem of state-space realization for 2-D systems from very different points of view.

In [4], Fornasini and Marchesini were using the algebraic point of view of Nerode equivalence. In this framework, the state space arises from the factorization of the 2-D input–output map. Fornasini and Marchesini were also the first to realize that a major difference between 1-D and 2-D systems is that we can introduce a *global state* and a *local state* in the 2-D case. The global state (which is of infinite dimension in general) preserves all the past information while the local state gives us the size of the recursions to be performed at each step by the 2-D filter.

However, Fornasini and Marchesini failed to exploit fully the structure of the global state and its relation to the local state, so that the state-space model they introduced is unsatisfactory, as we shall see, in the sense that what they introduce as the state is really only a "partial state" (as defined by Wolovich in [6] for 1-D systems). Indeed, this partial state does not obey a first-order difference equation (the notion of first order difference equation for linear systems on partially ordered sets has been defined by Mullans and Elliott in [7]). Attasi's model suffers from the same drawback.

On the other hand, Givone and Roesser in [8] and [5] have used a "circuit approach" to the problem of state space realization for 2-D systems. They present a model in which the local state is divided into an *horizontal* and a *vertical* state which are propagated, respectively, horizontally and vertically by *first-order* difference equations. From this point of view, the global state appears as the boundary condition necessary to propagate the state-space equations.

However, Givone and Roesser did not provide much motivation for the introduction of such a model and seemed unaware of the full-circuit interpretation of their model since they were not able to implement an arbitrary 2-D transfer function, say $H(z, \omega) = b(z, \omega)/a(z, \omega)$ (where $a(z, \omega)$ is assumed to be monic), with their state-space model.

Mitra *et al.*, gave an answer to this problem in [9] by presenting an implementation method for 2-D transfer functions using some delay elements z^{-1} and ω^{-1}. We shall see below that this approach is consistent with Roesser's model.

In this paper, we reconcile the algebraic and the circuit approach. In Appendix AI we study the properties of the global state and of its relation to the local state. We show that Roesser's model appears naturally as a way to describe the local state properties. For a transfer function $H(z, \omega) = b(z, \omega)/a(z, \omega)$ where $\delta_z a = n$, $\delta_\omega a = m$, we exhibit some canonical state-space forms (e.g., the controllability, observability, and controller forms).

However, these realizations are of size $nm + n + m$ and, unlike in the 1-D case, are not minimal, but by a simple reduction procedure, we can manage to reduce them to another simple canonical realization of size $n + 2m$ (respectively $2n + m$).

From the circuit point of view we then prove that there is one-to-one correspondence between Roesser's state-space realization and the implementations of a 2-D transfer function with delay elements z^{-1} and ω^{-1}: the number of delay elements will give us the size of the local state. Also the horizontal states (respectively, the vertical states) will be interpreted as the outputs of the z^{-1} delay elements (respectively, of the ω^{-1} delay elements).

This interpretation will enable us in Section III to present an easy $n + 2m$ (respectively $2n + m$) circuit implementation of the 2-D transfer function $H(z, \omega)$ which turns out to be exactly the same as the $n + 2m$ realization derived from Nerode equivalence.

All this strongly indicates that Roesser's model is the most general 2-D state-space model, and in fact we shall also prove in this section that the models of Attasi and Fornasini–Marchesini can be imbedded in Roesser's model, so that there will be no loss of generality in considering Roesser's model in the subsequent sections of this paper.

Roesser's model is the following:

$$x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \tag{2.1}$$

where x is the *local state*, x^h, an n-vector, is the *horizontal* state, x^v, an m-vector, is the *vertical* state, and

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}}_{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u(i,j)$$

$$y(i,j) = \underbrace{[C_1 \quad C_2]}_{C} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \qquad i,j \geqslant 0 \tag{2.2}$$

where u, the input, is an l-vector and y, the output, is a p-vector. Clearly x^h, the horizontal state, is propagated horizontally, and x^v, the vertical state is propagated vertically by first-order difference equations.

Now, if we take the (z, ω) transform of (2.2) (see Appendix A2)

$$\left[\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A\right] x(z,\omega) = B\,u(z,\omega) + \begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} \begin{bmatrix} x^h_{\delta_u}(\omega) \\ x^v_{\delta_h}(z) \end{bmatrix} \tag{2.3}$$

$$y(z,\omega) = C\,x(z,\omega)$$

so that the initial conditions are given by $x^h(0,j)$ and $x^v(i,0)$, $i, j \in \mathfrak{N}$. Then, let $\mathfrak{X}^h_{(0,0)} = \{x^h(0,j), j \in \mathfrak{N}\}$: $\mathfrak{X}^h_{(0,0)}$ belongs to $\mathfrak{X}^h$ the space of n vector sequences.

Similarly, $\mathfrak{X}^v_{(0,0)} = \{x^v(i,0), i \in \mathfrak{N}\}$ belongs to $\mathfrak{X}^v$ the space of m vector sequence and $\mathfrak{X}_{(0,0)} = (\mathfrak{X}^h_{(0,0)}, \mathfrak{X}^v_{(0,0)})$ belongs to $\mathfrak{X} = \mathfrak{X}^h \times \mathfrak{X}^v$ the *global state-space* of the 2-D system being considered. It is infinite dimensional and $\mathfrak{X}_{(0,0)}$ is the initial

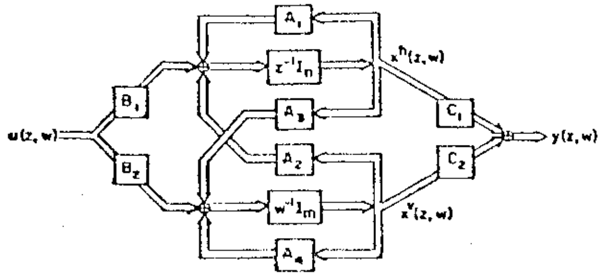


Fig. 1.

condition necessary to propagate the state-space equation (2.2).

We shall subsequently make a distinction between two types of initial conditions: for a *weak* (*local*) *initial condition* we assume $x(0, 0) = \xi$ arbitrary and $x^h(0, j) = x^v(i, 0) = 0$, for $i, j \geq 0$ for a *strong* (*global*) *initial condition* we assume that $\mathfrak{X}_{(0,0)} = \Xi$ is arbitrary.

Now, from the following circuit implementation (Fig. 1) of

$$H(z, \omega) = C\left[\begin{pmatrix} zI_n & \\ & \omega I_m \end{pmatrix} - A\right]^{-1} B$$

we can write directly the set of equations (2.3).

From there it is clear that there is a one-to-one correspondence between Roesser's model and circuit implementations with delay elements z^{-1} and ω^{-1}. It should be noted that this structure also arises naturally for delay-differential systems (see [10]).

Fornasini–Marchesini's model can be described as follows:

$$\begin{aligned} x(i+1, j+1) &= A_1\, x(i, j+1) + A_2\, x(i+1, j) \\ &\quad + A_0\, x(i, j) + B\, u(i, j) \\ y(i, j) &= C\, x(i, j), \quad i, j \geq 0 \end{aligned} \tag{2.4}$$

where x is an m-vector, y a p-vector, and u an l-vector. Attasi's model corresponds to the special case when $A_0 = -A_1 A_2 = -A_2 A_1$.

From the fact that $x(i+1, j+1)$ depends on $x(i, j)$ it is clear that (2.4) are not first-order difference equations so that x is only a "partial state." To be more precise let us define

$$\xi(i, j) = x(i, j+1) - A_2\, x(i, j) \tag{2.5}$$

so that

$$\xi(i+1, j) = A_1\, x(i, j+1) + A_0\, x(i, j) + B\, u(i, j)$$

Hence

$$\begin{bmatrix} \xi(i+1, j) \\ x(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_0 + A_1 A_2 \\ I_n & A_2 \end{bmatrix} \begin{bmatrix} \xi(i, j) \\ x(i, j) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(i, j) \tag{2.6}$$

$$y(i, j) = [0 \quad C] \begin{bmatrix} \xi(i, j) \\ x(i, j) \end{bmatrix}$$

is a way to recast Fornasini–Marchesini's model in Roesser's form. From (2.6) it is clear that $x(i, j)$ is only a partial state, the full state being

$$\begin{bmatrix} \xi(i, j) \\ x(i, j) \end{bmatrix}.$$

A consequence is that the properties of controllability and observability considered by Fornasini–Marchesini in [4] or by Attasi in [2] are not adequate since they involve only the partial state x.

III. Circuit Realizations and Hardware Designs

First, we can note that the notion of "dynamic elements," "multipliers" and "adders" is at the center of circuit theory. In the 1-D discrete-time case, the dynamic elements used are (time) delay elements. The 1-D realization problems have been well studied and, given any transfer function, it is well known that the realization can be readily found in certain standard (e.g., controller canonical) forms [11]. For the realization of a 2-D transfer function, a major difference is that two types of dynamic elements are needed:

$$x(z^{-1}, \omega^{-1}) \longrightarrow \boxed{z^{-1}} \longrightarrow z^{-1}\, x(z^{-1}, \omega^{-1})$$

$$x(z^{-1}, \omega^{-1}) \longrightarrow \boxed{\omega^{-1}} \longrightarrow \omega^{-1}\, x(z^{-1}, \omega^{-1})$$

They will be termed as "horizontal delay element" (z^{-1}) and "vertical delay element" (ω^{-1}), respectively.

Now an important problem is that of how to use 2-D dynamic elements, multipliers and adders, to realize a 2-D digital filter with the transfer function:

$$H(z^{-1}, \omega^{-1}) = \frac{b(z^{-1}, \omega^{-1})}{a(z^{-1}, \omega^{-1})} = \frac{\sum_{i=0}^{n} \sum_{j=0}^{m} b_{ij}\, z^{-i}\, \omega^{-j}}{\sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij}\, z^{-i}\, \omega^{-j}}. \tag{3.1}$$

We can do this in two steps. First we rewrite (3.1) in a rational-gain representation, i.e.,

$$H(z^{-1}, \omega^{-1}) = \frac{\sum_{i=0}^{n} b_i(\omega^{-1})\, z^{-i}}{\sum_{i=0}^{n} a_i(\omega^{-1})\, z^{-i}}. \tag{3.2}$$

Without loss of generality, we can assume $a_{00} = 1$ and we denote

$$a_0(\omega^{-1}) \triangleq 1 + \tilde{a}_0(\omega^{-1})$$

Thus using the 1-D realization technique, we write down the realization shown in Fig. 2, where the gains of the multipliers are represented in $F[\omega^{-1}]$.

The realization is almost achieved: in addition to the n horizontal delay elements of Fig. 2 we need only m vertical delay elements to implement the feedback gains $\{a_i(\omega^{-1}),\, i = 0, 1, \cdots, m\}$ and m other vertical delay elements to implement the readout gains $\{b_i(\omega^{-1}),\, i = 0, 1, \cdots, m\}$. Thus, the complete realization shown in Fig. 3 requires only $n + 2m$ dynamic elements.

This realization is a standard (canonical) one; its structure is very simple and it involves only real gains. Note also that we need fewer dynamic elements than was suggested by the implementations of [9].

A. State-Space Model Representation

As remarked in Section II, circuit implementations with delay elements z^{-1} and ω^{-1} are in a one-to-one correspondence with state-space models of Roesser's type. The outputs of the z^{-1} delays are the horizontal states and the outputs of the ω^{-1} delays are the vertical states.

Thus, the implementation of Fig. 3 can be transformed readily into the following state-space model:

$$\begin{array}{l} n\{ \\ m\{ \\ m\{ \end{array}\begin{bmatrix} x_h(i+1,j) \\ x_{v_1}(i,j+1) \\ x_{v_2}(i,j+1) \end{bmatrix} = A \begin{bmatrix} x_h(i,j) \\ x_{v_1}(i,j) \\ x_{v_2}(i,j) \end{bmatrix} + b\, u(i,j) \qquad (3.3)$$

$$y(i,j) = c\, x(i,j)$$

where

$$b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \cdots \\ a_{01} \\ \vdots \\ a_{0m} \\ \cdots \\ b_{01} \\ \vdots \\ b_{0m} \end{bmatrix} \qquad c = [\tilde{b}_{10} \cdots \tilde{b}_{n0} \vdots -b_{00}\ 0 \cdots 0 \vdots 1\ 0 \cdots 0]$$

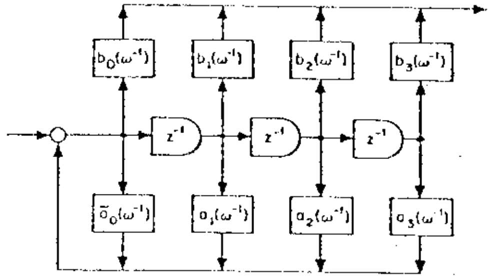


Fig. 2.

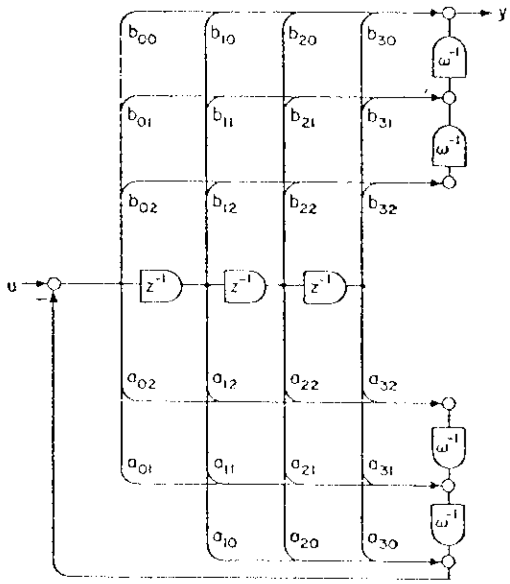


Fig. 3. 2-D controller form realization.

$$A = \left[\begin{array}{cccc|cccc|cccc}
-a_{10} & -a_{20} & \cdots & -a_{no} & -1 & & & & & & & \\
1 & & & & & & & & & & & \\
& \ddots & & \bigcirc & & \bigcirc & & & & \bigcirc & & \\
\bigcirc & & \ddots & 0 & & & & & & & & \\
& & & 1 & & & & & & & & \\ \hline
\tilde{a}_{11} & & & \tilde{a}_{n1} & -a_{01} & 1 & & & & & & \\
& & & & \vdots & & \ddots & \bigcirc & & & & \\
& \tilde{a}_{ij} & & & \vdots & & & & & \bigcirc & & \\
& & & & \vdots & \bigcirc & & 1 & & & & \\
\tilde{a}_{1m} & & & \tilde{a}_{nm} & -a_{0m} & & & 0 & & & & \\ \hline
\tilde{b}_{11} & & & \tilde{b}_{n1} & -b_{01} & & & & 0 & 1 & & \\
& & & & \vdots & & & & & \ddots & \bigcirc & \\
& \tilde{b}_{ij} & & & \vdots & & \bigcirc & & & & & \\
& & & & \vdots & & & & \bigcirc & & \ddots & 1 \\
\tilde{b}_{1m} & & & \tilde{b}_{nm} & -b_{0m} & & & & & & & 0
\end{array}\right]$$

with

$$\tilde{a}_{ij} = a_{ij} - a_{i0}\, a_{0j}, \qquad 1 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant m$$
$$\tilde{b}_{ij} = b_{ij} - a_{i0}\, b_{0j}, \qquad 1 \leqslant i \leqslant n, \quad 0 \leqslant j \leqslant m.$$

We shall postpone the discussion of some interesting properties of 2-D state-space models to later sections. We shall explore now another phase of the realization problem—hardware implementation.

B. Hardware Design of 2-D Digital Filter

The idea of using two types of dynamic elements is not very abstract; it is very natural in delay-differential systems. However, before considering its practical application to image systems, two remarks have to be made:

1) Because the "spatial" dynamic elements seem unimplementable, we need to replace them by time-delay elements.

2) In order to have a finite order description, we shall only consider a bounded frame system, i.e., we assume that the picture frame of interest is an $M \times N$ frame (with vertical width M and horizontal length N).

Note that in order to use time delay elements we need first to find a way to code a 2-D spatial system into a 1-D (discrete-time) system and vice versa. Thus we shall propose the following system, composed of three subsystems in series;

i) The *input scan generator* codes the 2-D spatial input into 1-D (time) data according to the mapping function $t(\cdot,\cdot)$

$$t(i,j) = iM + jN \tag{3.4}$$

where M and N are relatively prime integers.

ii) A *1-D (discrete-time) digital filter* processes the 1-D data generated by (i). This subsystem is implemented by replacing z^{-1} by δ, ω^{-1} by Δ in a 2-D circuit realization (e.g., 2-D controller form). δ and Δ are chosen as

$$\delta = D^M = M\text{-units delay element}$$
$$\Delta = D^N = N\text{-units delay element.} \tag{3.5}$$

iii) The *output frame generator* decodes the 1-D (discrete-time) output of the 1-D digital filter described above into a 2-D (discrete-spatial) picture according to the inverse mapping of (3.4).

$$(i(t), j(t)) = (Pt \bmod N, [t - (Pt \bmod N) M]/N) \tag{3.6}$$

where P is the unique integer such that

$$PM - QN = 1 \quad \text{and} \quad 0 < P < N \tag{3.7}$$

The overall design is shown in Fig. 4.

We shall show that this proposed 2-D digital filter will work. Let us note the 1-D (discrete-time) output will be

$$\begin{aligned}\mathcal{Y}(D) &= \mathcal{H}(D)\,\mathcal{U}(D)\\ &= H(z^{-1},\omega^{-1})\,u(z^{-1},\omega^{-1})\Big|_{z^{-1}=D^M,\,\omega^{-1}=D^N}\\ &= \sum_i \sum_j y_{i,j}\, z^{-i}\,\omega^{-j}\Big|_{z^{-1}=D^M,\,\omega^{-1}=D^N}\end{aligned} \tag{3.8}$$

where $\{y_{ij}\}$ represents the 2-D (discrete spatial) output data field. Note also that

$$\mathcal{Y}(D) \triangleq \sum_t \mathcal{Y}_t\, D^{-t} \tag{3.9}$$

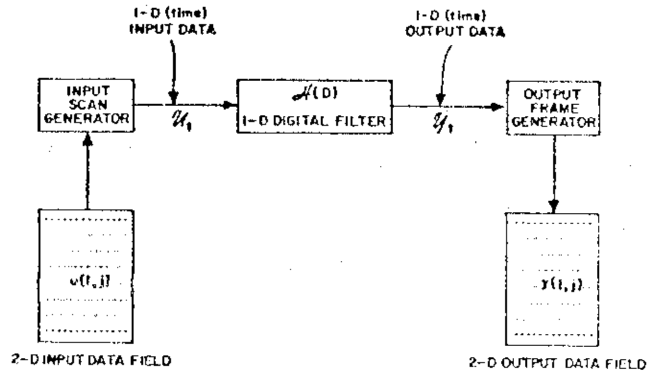


Fig. 4. Overall design of 2-D digital filter.

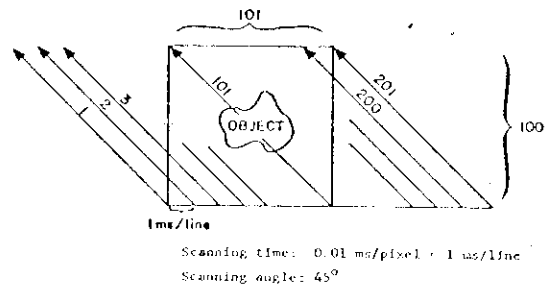


Fig. 5. Input scan generator and output frame generator.

Comparing (3.8) and (3.9), it is clear that

$$\mathcal{Y}_t = \sum_{(i,j):\, iM+jN=t} y_{i,j}. \tag{3.10}$$

Since the system is a causal system

$$y_{i,j} = 0, \qquad \text{if} \quad i,j < 0. \tag{3.11}$$

Let us consider only the integer t with

$$t = iM + jN, \qquad i < N,\ j < M$$

then (3.10) and (3.11) give

$$\mathcal{Y}_t = y_{i,j}$$

since, for this special case, the summation set of (3.10) contains only one nonzero point. Therefore, we will obtain a *bona fide* output picture inside the $M \times N$ frame.

Let us remark that the image scanning and display systems used in our 2-D digital filter design are not as complicated as they look.

In fact, they can be very simple as we shall show in the next design example.

Example: Problem—Design a 2-D digital filter for

$$H(z^{-1},\omega^{-1}) = \frac{1}{1 + 0.2z^{-1} + 0.3\omega^{-1} + 0.1z^{-1}\omega^{-1}}$$

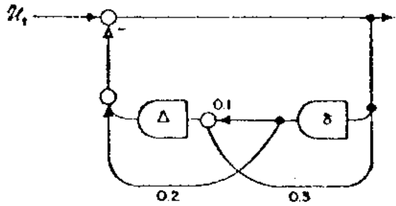
for a frame: $M \times N = 100 \times 101$. Assume $D = 0.01$ ms.

Solution:

(i) *ISG*—In this special frame (with $N = M + 1$), the input scanning generator is indeed very simple, as shown in Fig. 5.

(ii) *1-D Digital Filter*—Constructing the 2-D realization of Fig. 3 and then replacing z^{-1} by δ and ω^{-1} by Δ we have the 1-D realization shown in Fig. 6.

(iii) *OFG*—The output frame generator does the reverse of the ISG, displaying the picture instead of scanning.



Δ: 1.01 ms delay element
δ: 1.00 ms delay element

Fig. 6.

C. *Dimensionality of Global State*

Considering a bounded frame ($M \times N$) system, it is interesting to know the dimension of the *global state* (or initial conditions) needed to process the $M \times N$ "future" data field. Since vertical states convey information vertically, all the vertical states along the X-axis are necessary initial conditions and their dimension is mN. Similarly, all the horizontal states along the Y-axis are necessary initial conditions (with dimension nM) since they convey information horizontally. Therefore, in the bounded frame case a total number of $mN + nM$ are needed to summarize the "past" information.

This very same idea can be used again from a computational point of view. Indeed, the number of required storage elements for recursive computations is also equal to $nM + mN$ if initial conditions are not zero. However, it is quite often the case that the system starts with zero initial conditions; then the size of storage required is reduced to mN (respectively nM) which is used to store the updated data row by row (respectively column by column). No storage is needed for the rest of the initial conditions—nM horizontal states (respectively mN vertical states)—since they are assumed to be zero. This is consistent with the result of Read [12] derived from a direct polynomial approach.

Another interesting observation concerns the dimension of the 1-D digital filter contained by our 2-D digital filter design discussed above. Since it needs nM-unit-delays and mN-unit-delays, the corresponding 1-D state-space also has a dimension equal to $nM + mN$. Note that, despite the high dimension of the corresponding 1-D filter, its high sparsity is very encouraging for further studies.

In short, our studies on the dimensionality of 2-D global states have reached a consistent conclusion from either theoretical or practical approaches.

IV. Local and Global Controllability (Observability) of 2-D Systems

In the study of 1-D systems, the notions of controllability and observability were very fruitful, by themselves, since they yielded a canonical decomposition of the state-space and also because a state-space realization was minimal (i.e., of minimum size) iff it was controllable and observable.

Such properties would be also desirable for 2-D systems, and in [5] and [13] Givone and Roesser introduced what we shall call *local* controllability and observability notions. However, unlike what these authors seemed to believe in [5], these notions are not very satisfactory in the sense that a state-space model can be locally controllable and observable without being minimal, and conversely a system can be minimal without being locally controllable or locally observable, as we shall see. It should also be noted that the controllability and observability notions introduced by Attasi in [2] or Fornasini-Marchesini in [4] were also local notions and had the additional disadvantage, as we remarked earlier, of considering such properties only for a partial state.

Definition 4.1: a) the system (2.2) is locally controllable if, for zero initial conditions ($\mathcal{X}_{(0,0)}) \equiv 0$ and ξ an arbitrary $n + m$ vector, there exists $N, M > 0$ and a sequence of inputs $u(i, j)$ $(0, 0) \leqslant (i, j) < (N, M)$ such that $x(N, M) = \xi$. b) The system (2.2) is locally observable if there is no nonzero weak (local) initial condition such that for zero input $u(i, j) \equiv 0$ $i, j \geqslant 0$ the output is also identically zero: $y(i, j) \equiv 0$ $i, j \geqslant 0$.

To be more precise, if we define the transition matrix $A^{(i,j)}$ by

$$\left[I_{n+m} - \begin{pmatrix} z^{-1} I_n & 0 \\ 0 & \omega^{-1} I_m \end{pmatrix} A\right]^{-1}$$

$$= \sum_{h \geqslant 0} \left[\begin{pmatrix} z^{-1} I_n & 0 \\ 0 & \omega^{-1} I_m \end{pmatrix} A\right]^h = \sum_{i,j \geqslant 0} A^{(i,j)} z^{-i} \omega^{-i}. \tag{4.1}$$

Then, from (2.3) it is clear that the solution of (2.2) is

$$x(i,j) = \sum_{k=0}^{j} A^{(i,j-k)} \begin{pmatrix} x^h(0,k) \\ 0 \end{pmatrix} + \sum_{r=0}^{i} A^{(i-r,j)} \begin{pmatrix} 0 \\ x^v(r,0) \end{pmatrix}$$
$$+ \sum_{(0,0) \leqslant (k,r) < (i,j)} [A^{(i-1-k,j-r)} B^{(1,0)}$$
$$+ A^{(i-k,j-r-1)} B^{(0,1)}] \, u(k,r) \tag{4.2}$$

where

$$B^{(1,0)} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \qquad B^{(0,1)} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

and if we take into account the 2-D Cayley–Hamilton theorem (see [13], [14]), it is clear from (4.2) that the system (2.2) is locally controllable iff

$$C_{n,m} = [M(1,0)\, M(0,1) \cdots M(i,j) \cdots M(n,m)] \tag{4.3}$$

is full rank, where $(0, 0) < (i, j) \leqslant (n, m)$ and

$$M(i,j) = A^{(i-1,j)} B^{(1,0)} + A^{(i,j-1)} B^{(0,1)}.$$

Similarly, the system (2.2) is locally observable iff

$$0_{n,m} = \begin{bmatrix} C \\ \cdot \\ \cdot \\ \cdot \\ CA^{(i,j)} \\ \cdot \\ \cdot \\ \cdot \\ CA^{(n,m-1)} \\ \cdot \\ \cdot \\ \cdot \\ CA^{(n-1,m)} \end{bmatrix} \tag{4.4}$$

is full rank, where $(0, 0) \leqslant (i, j) < (n, m)$.

These notions, introduced by Roesser, look like natural generalizations of the 1-D case. However they are not closely related to the notion of minimality, as will be shown in Examples 4.2 and 4.3. Example 4.3 will also illustrate the fact that these notions as presented in (4.3) and (4.4) do not yield a canonical decomposition of the state-space and that if we want to obtain such a decomposition we need to consider *sep-*

arately the local controllability (observability) of the horizontal and of the vertical state.

Example 4.2: Consider the transfer function $H(z, \omega) = (z - \omega)/(z\omega - 1)$. Then, using Nerode equivalence (see Appendix A1) we obtain the controller realization:

$$\begin{pmatrix} z.1 & 0 \\ 0 & \omega I_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\begin{array}{c:cc} 0 & 0 & 1 \\ \hdashline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = (-1 \;\; 1 \;\; 0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.5}$$

It is easy to check that $[M(1, 0)\, M(0, 1)\, M(1, 1)] = I_3$ and

$$\begin{bmatrix} C \\ CA^{(1,0)} \\ CA^{(0,1)} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

so that this realization is locally controllable and observable. But it is also easy to verify that

$$\begin{pmatrix} zx_1 \\ \omega x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u \qquad y = (1 \;\; 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is also a realization of $H(z, \omega)$ which is clearly minimal since we need at least one z^{-1} delay element and one ω^{-1} delay element to realize $H(z, \omega)$.

Hence the realization (4.3) is not minimal even though it is locally controllable and observable.

Example 4.3: Consider the state-space model

$$\begin{pmatrix} z.1 & 0 \\ 0 & \omega I_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{B} u$$

$$y = \underbrace{(1 \;\; 1 \;\; 1)}_{C} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{4.6}$$

The observability matrix is

$$\begin{bmatrix} C \\ CA^{(1,0)} \\ CA^{(0,1)} \\ CA^{(1,1)} \\ CA^{(0,2)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 1 & 3 \end{bmatrix}.$$

Its rank is 2, so that the system (4.6) is not locally observable and the rank of

$$[M(1,0)\, M(0, 1)\, M(1, 1)] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

is 3 so that the system is locally controllable. However,

$$H(z, \omega) = C \left[\begin{pmatrix} z.1 & 0 \\ 0 & \omega I_2 \end{pmatrix} - A \right]^{-1} B$$
$$= \frac{\omega(z + \omega - 1)}{(\omega - 1)(z\omega - z - \omega)} = \frac{b(z, \omega)}{a(z, \omega)}$$

so that there is no cancellation in the transfer function, and moreover since $\delta_z a = 1$, $\delta_\omega a = 2$ we need at least 1 horizontal state and 2 vertical states to realize $H(z, \omega)$ so that the system (4.6) is minimal in spite of the fact that it is not locally observable.

From Examples 4.2 and 4.3 we can conclude that the local controllability and observability notions are not adequate as far as minimality is concerned. However, by looking more closely at Example 4.3 it is easy to reformulate the local controllability and observability notions in a way that would guarantee at least that a minimal system is locally controllable and observable.

We can remark first that, in [13], Roesser introduces the following class of similarity transform:

$$\begin{pmatrix} \hat{x}^h \\ \hat{x}^v \end{pmatrix} = \underbrace{\begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix}}_{T} \begin{pmatrix} x^h \\ x^v \end{pmatrix}. \tag{4.7}$$

Then, if $\hat{B} = TB$, $\hat{A} = TAT^{-1}$, $\hat{C} = CT^{-1}$,

$$\hat{C} \left[\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A \right]^{-1} \hat{B} = C \left[\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A \right]^{-1} B$$

so that the transfer function is invariant, while this would not be the case for a more general

$$T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}.$$

But we can observe that the definitions (4.3) and (4.4) for local controllability and observability are not compatible with the class of similarity transform, since a system can be noncontrollable (nonobservable) and at the same time we may not be able to find a similarity transform of the class (4.7) that would display the noncontrollable (nonobservable) parts of the state. Indeed we are not sure that if $\xi \in \operatorname{Ker} 0_{n,m}$ ξ is either of the form

$$\begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \begin{matrix} \}\, n \\ \}\, m \end{matrix} \qquad \text{or} \qquad \begin{pmatrix} 0 \\ \xi_2 \end{pmatrix} \begin{matrix} \}\, n \\ \}\, m \end{matrix}$$

as would be required by (4.7). This is what occurs in Example 4.3 since

$$\operatorname{Ker} 0_{1,2} = \begin{pmatrix} 1 \\ \hdashline -1 \\ 0 \end{pmatrix}$$

has nonzero horizontal and vertical components.

However this suggests the following reformulation of local controllability and observability notions. Let

$$0_{n,m} = [\overbrace{0^h_{n,m}}^{n} \;\vdots\; \overbrace{0^v_{n,m}}^{m}] \qquad C = \begin{bmatrix} C^h_{n,m} \\ \cdots\cdots \\ C^v_{n,m} \end{bmatrix} \begin{matrix} \}\, n \\ \\ \}\, m \end{matrix}$$

Proposition 4.4: x^h and x^v are separately locally controllable (respectively locally observable) iff C^h and C^v (respectively 0^h and 0^v) are separately full rank.

This new formulation is clearly compatible with the class (4.7) of similarity transforms and yields a canonical decomposition of the horizontal and vertical state. As a consequence, if a system is minimal, the horizontal and vertical states are separately locally controllable and observable.

As an illustration, the system of Example 4.3 is locally observable (separately) since the matrices

$$0_h = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad 0_v = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \\ 1 & 3 \end{pmatrix} \text{are full rank.}$$

We present now some alternative controllability and observability concepts.

In the 1-D case, there was a close connection between controllability and observability properties and the properties of surjectivity and injectivity of the maps appearing in the state-space factorization of the input–output map (see [15]).

The same approach for 2-D systems yields the following global controllability and observability notions (i.e., we require controllability or observability of the global state).

Let $\mathfrak{U}$ be the space of past inputs (as defined in the Appendix A1), $\mathfrak{Y}$ the space of future outputs, $\mathfrak{X} = \mathfrak{X}^h \times \mathfrak{X}^v$ the global state. Then $\mathcal{C}$, the *controllability* map is as follows:

$$\mathcal{C}: u(\cdot\,,\cdot) \in \mathfrak{U} \longrightarrow \mathfrak{X}_{(0,0)} \in \mathfrak{X}$$

where

$$\mathfrak{X}_{(0,0)} = ((x^h(0,j), j \in \mathfrak{N}),\ (x^v(i,0), i \in \mathfrak{N}))$$

and

$$x^h(0,j) = I^{(1,0)} \sum_{(-N,-M) \leq (k,r) < (0,j)} M(-k, j-r)\, u(k,r)$$

$$x^v(i,0) = I^{(0,1)} \sum_{(-P,-Q) \leq (k,r) < (i,0)} M(i-k, -r)\, u(k,r)$$

$$I^{(1,0)} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \quad I^{(0,1)} = \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix}. \tag{4.8}$$

Similarly θ, the *observability* map is

$$\theta: \mathfrak{X}_{(0,0)} \in \mathfrak{X} \longrightarrow y_{(\cdot,\cdot)} \in \mathfrak{Y}$$

with

$$y(i,j) = \sum_{r=0}^{j} CA^{(i,j-r)} \begin{pmatrix} x^h(0,r) \\ 0 \end{pmatrix} + \sum_{k=0}^{i} CA^{(i-k,j)} \begin{pmatrix} 0 \\ x^v(k,0) \end{pmatrix}. \tag{4.9}$$

Definition 4.5:

(i) The system (2.2) is globally controllable iff $\mathcal{C}$ is a surjective map.

(ii) It is globally observable iff θ is an injective map.

However, as far as minimality is concerned the global concepts are as unsatisfactory as the local ones, since a system can be minimal but not globally observable.

Example 4.6: The system

$$\begin{pmatrix} zx_1 \\ \omega x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$

$$y = (1 \quad 1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

of Example 4.2 is minimal. Yet, it is not globally observable since the global initial condition

$$x^h(0,j) = 1, \quad \text{for all} \quad j \in \mathfrak{N}$$

$$x^v(i,0) = -1, \quad \text{for all} \quad i \in \mathfrak{N}$$

is not observable, indeed if $u \equiv 0$ then

$$x(i,j) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{for all} \quad i,j$$

$$y(i,j) = 0, \quad \text{for all} \quad i,j.$$

Hence the most natural generalizations of the 1-D approach to controllability and observability fail in the 2-D case. In the next section, we present an alternative definition based on coprimeness notions. We also give an interpretation of the failure of global controllability (observability).

V. Modal Controllability (Observability) and Minimality

In the 1-D case, the relative primeness concepts could also be used to define controllability and observability. In [16] Rosenbrock proved that

A, B was controllable iff $zI - A, B$ were left coprime

C, A was observable iff $C, zI - A$ were right coprime.

This approach can be generalized very easily to 2-D systems and will also provide a definition of minimality.

Definition 5.1: Let $H(z, \omega) = VT^{-1}U$ where V, T, U are 2-D polynomial matrices. It is a *minimal* description of $H(z, \omega)$ iff

V, T are right coprime and T, U are left coprime.

This amounts to requiring that there is no cancellation in the 2-D transfer function $H(z, \omega)$. In part I we also proved the important property that if (V, T, U) and (V_1, T_1, U_1) are two minimal descriptions of H, $|T| = |T_1|$. We also presented an algorithm to extract the greatest common right (left) divisor of two polynomial matrices, which enables us to find a minimal description of H from a nonminimal one.

Thus in the state-space description case

$$H = C\left[\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A\right]^{-1} B$$

is minimal iff

$$\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A, B \quad \text{are left coprime} \tag{5.1}$$

and

$$C, \begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A \quad \text{are right coprime.} \tag{5.2}$$

Definition 5.2:

(i) A, B is *modally* controllable if (5.1) holds.
(ii) C, A is *modally* observable if (5.2) holds.

These definitions are clearly connected to minimality but the state-space significance of controllability and observability disappears in this formulation. This is why we shall give now an equivalent state-space characterization of the notions of modal controllability and observability. Another consequence is that for a single-input–single-output system, if

$$H(z,\omega) = \frac{b(z,\omega)}{a(z,\omega)}$$

and if b and a are coprime with $\delta_z a = n\ \delta_\omega a = m$, then if

$$C\left[\begin{pmatrix} zI_p & 0 \\ 0 & \omega I_q \end{pmatrix} - A\right]^{-1} B$$

is a minimal realization of $H(z,\omega)$ we must have

$$\left|\begin{pmatrix} zI_p & \\ & \omega I_q \end{pmatrix} - A\right| = a(z,\omega)$$

and hence $p = n$ and $q = m$.

Hence the validity of our definition of minimality of a state-space model will depend on our ability to realize a transfer function of order (n, m) with $n + m$ states. This problem will be considered in Section VI.

A consequence of the relative primeness criterion for 2-D polynomial matrices given in Part I is that C and

$$\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A$$

are right coprime iff

$$\rho\begin{bmatrix} \begin{pmatrix} \xi_1 I_n & 0 \\ 0 & \xi_2 I_m \end{pmatrix} - A \\ C \end{bmatrix} = n + m$$

for any generic point (ξ_1, ξ_2) of any irreducible algebraic curve V_i appearing in the decomposition of V, the algebraic curve defined by

$$\left|\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A\right| = 0.$$

It is to be noted that the rank is considered over the field $K(\xi_1, \xi_2)$.

Now, assume that

$$C \text{ and } \begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A$$

are not right coprime, this means that there is an irreducible algebraic curve W, a generic point (ξ_1, ξ_2) of this curve and $p_h \in K^n(\xi_1, \xi_2)$, $p_v \in K^m(\xi_1, \xi_2)$ such that

$$A\begin{pmatrix} p_h \\ p_v \end{pmatrix} = \begin{pmatrix} \xi_1\, p_h \\ \xi_2\, p_v \end{pmatrix}$$

$$C\begin{pmatrix} p_h \\ p_v \end{pmatrix} = 0. \tag{5.3}$$

Then, if we define

$$\mathcal{A}^{(1,0)} = \begin{bmatrix} A_1 & A_2 \\ 0 & \xi_1 I_m \end{bmatrix} = A^{(1,0)} + \xi_1 I^{(0,1)}$$

and

$$\mathcal{A}^{(0,1)} = \begin{bmatrix} \xi_2 I_n & 0 \\ A_3 & A_4 \end{bmatrix} = A^{(0,1)} + \xi_2 I^{(1,0)} \tag{5.4}$$

the set of equations (5.3) can be rewritten

$$\mathcal{A}^{(1,0)}\begin{pmatrix} p_h \\ p_v \end{pmatrix} = \xi_1\begin{pmatrix} p_h \\ p_v \end{pmatrix},$$
$$\mathcal{A}^{(0,1)}\begin{pmatrix} p_h \\ p_v \end{pmatrix} = \xi_2\begin{pmatrix} p_h \\ p_v \end{pmatrix} \tag{5.5}$$
$$C\begin{pmatrix} p_h \\ p_v \end{pmatrix} = 0$$

so that $\begin{pmatrix} p_h \\ p_v \end{pmatrix}$

is a common eigenvector of $\mathcal{A}^{(1,0)}$ and $\mathcal{A}^{(0,1)}$ which is in the null space of C. Define the transition matrix

$$\mathcal{A}^{(i,j)} = \mathcal{A}^{(1,0)}\ \mathcal{A}^{(i-1,j)} + \mathcal{A}^{(0,1)}\ \mathcal{A}^{(i,j-1)}. \tag{5.6}$$

From (5.5) and (5.6) it is easy to prove by recurrence that

$$\mathcal{A}^{(i,j)}\begin{pmatrix} p_h \\ p_v \end{pmatrix} = \frac{(i+j)!}{i!\,j!}\,\xi_1^i\,\xi_2^j\begin{pmatrix} p_h \\ p_v \end{pmatrix}, \qquad \text{for all } i, j.$$

Hence

$$C\,\mathcal{A}^{(i,j)}\begin{pmatrix} p_h \\ p_v \end{pmatrix} = 0, \qquad \text{for all } i, j. \tag{5.7}$$

But this is equivalent to

$$\sum_{r=0}^{j} CA^{(i,j-r)}\begin{pmatrix} \xi_2^r\, p_h \\ 0 \end{pmatrix} + \sum_{k=0}^{i} CA^{(i-k,j)}\begin{pmatrix} 0 \\ \xi_1^k\, p_v \end{pmatrix} = 0. \tag{5.8}$$

From (5.4) and (5.5) it can indeed be proved by recurrence that

$$C\mathcal{A}^{(i,j)}\begin{pmatrix} p_h \\ p_v \end{pmatrix} = \sum_{r=0}^{j} CA^{(i,j-r)}\begin{pmatrix} \xi_2^r\, p_h \\ 0 \end{pmatrix} + \sum_{k=0}^{i} CA^{(i-k,j)}\begin{pmatrix} 0 \\ \xi_1^k\, p_v \end{pmatrix}.$$

A consequence of (5.8) is that

$$\mathfrak{X}(\xi_1, \xi_2) = ((\xi_2^r\, p_h, r \in \eta), (\xi_1^k\, p_v, k \in \eta)) \in \operatorname{Ker} \mathcal{O} \tag{5.9}$$

where $\mathcal{O}$ is the global observability operator defined by (4.9).

So, if the system is *not modally observable*, what happens is that a full *cone* (5.9) associated with the algebraic variety W and the generic point (ξ_1, ξ_2) is *in the null space of the observability operator* $\mathcal{O}$.

Equation (5.9) provides also some motivation for the qualification "modal" in our observability definition. In the 1-D case, a system was not observable iff there was a mode λ_0 and an associated eigenvector $p(\lambda_0)$ which was not observable (i.e., in the null-space of the observability matrix). In the 2-D case, the modes become irreducible algebraic curves $W(\xi_1, \xi_2)$ with which are associated *eigencones* as in (5.9), and a system is not observable if one of these eigencones is in the null-space of the observability operator $\mathcal{O}$.

To be more precise, the eigencone defined by (5.9) in the global space $\mathfrak{X}$ is such that

$$x(i,j) = \frac{(i+j)!}{i!\,j!}\,\xi_1^i\,\xi_2^j \begin{pmatrix} p_h(\xi_1,\xi_2) \\ p_v(\xi_1,\xi_2) \end{pmatrix}, \qquad \text{for all } i, j$$

which is the 2-D generalization of

$$x(i) = \lambda_0^i\, p(\lambda_0), \qquad \text{for all } i$$

(where we suppose that $u \equiv 0$, $\mathfrak{X}_{(0,0)}$ being defined by (5.9) in the 2-D case, $x(0) = p(\lambda_0)$ in the 1-D case).

Remark: In Part 1, we had remarked that algebraic varieties of dimension 1 and not of dimension 0 were to be considered for the 2-D coprimeness of two matrix polynomials $A(z,\omega)$ and $B(z,\omega)$. Similarly, for the notion of global observability eigencones associated with algebraic varieties of dimension 1 are to be considered and not those associated with algebraic varieties of dimension 0.

This gives us an interpretation of why global observability notions fail. Indeed, for Example 4.6

$$\begin{bmatrix} \begin{pmatrix} z.1 & \\ & \omega.1 \end{pmatrix} - A \\ C \end{bmatrix} = \begin{bmatrix} z & 1 \\ 1 & \omega \\ 1 & 1 \end{bmatrix}$$

is not full rank at the point $(z_1, \omega_1) = (1, 1)$ which is an algebraic variety of dimension 0, and

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$(1 \quad 1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

defines an eigencone of dimension 0, which, as we observed in Example 4.6, is included in the null-space of the observability map σ.

To fix ideas let us consider the following example.

Example 5.3: We saw in Example 4.2 that the system

$$\begin{pmatrix} z.1 & 0 \\ 0 & \omega I_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\begin{array}{c:cc} 0 & 0 & 1 \\ \hdashline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$

$$y = (-1 \quad 1 \quad 0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

was not minimal, though it was locally controllable and observable. Now

$$\det \begin{pmatrix} z.1 & 0 \\ 0 & \omega I_2 \end{pmatrix} - A = \omega(z\omega - 1)$$

so that this system has two modes (eigencurves):

V_1 defined by $\omega = 0$, $\quad z$ arbitrary;

V_2 defined by $(\xi_1, \xi_2) = (z, 1/z)$.

$$\rho \left[\begin{array}{ccc:c} z & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{array}\right] = 2, \qquad \text{so that } V_1 \text{ is not modally controllable.}$$

If we test the modal controllability of V_2

$$\rho \left[\begin{array}{ccc:c} z & 0 & -1 & 1 \\ 0 & \frac{1}{z} & -1 & 1 \\ -1 & 0 & \frac{1}{z} & 0 \end{array}\right] = 3, \qquad \text{so that } V_2 \text{ is controllable.}$$

Similarly, testing the modal observability

$$V_1: \qquad \rho \begin{bmatrix} z & 0 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} = 3$$

$$V_2: \qquad \rho \begin{bmatrix} z & 0 & -1 \\ 0 & \frac{1}{z} & -1 \\ -1 & 0 & \frac{1}{z} \\ -1 & 1 & 0 \end{bmatrix} = 3$$

so that the system is modally observable.

Then, we can clarify the notion of eigencone. First, by duality, the observability properties of (B', A') are the same as the controllability properties of (A, B); this implies that V_1 is not modally observable for the pair $\overline{C} = B' = (1 \quad 1 \quad 0)$

$$\overline{A} = A' = \left(\begin{array}{c:cc} 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right).$$

(This duality argument avoids the difficulty of defining left eigencones associated with A).

Now, since

$$\begin{bmatrix} z & 0 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\overline{A}\left[\begin{array}{c} 1 \\ \hdashline -1 \\ z \end{array}\right] = \begin{bmatrix} z \cdot 1 \\ 0 \begin{pmatrix} -1 \\ z \end{pmatrix} \end{bmatrix} \qquad \text{and} \qquad \overline{C}\begin{bmatrix} 1 \\ -1 \\ z \end{bmatrix} = 0.$$

And, if we consider the global initial condition

$$x_1(0, j) = 0, \qquad \text{for all } j\text{'s}$$

$$\begin{pmatrix} x_2(i,0) \\ x_3(i,0) \end{pmatrix} = z^i \begin{pmatrix} -1 \\ z \end{pmatrix}, \qquad \text{for all } i\text{'s} \tag{5.10}$$

when z varies, this initial condition describes a cone in the global space $\mathfrak{X}$. This cone has the property that if we assume $u(i,j) \equiv 0$

$$x(i,j) = \begin{cases} z^i \begin{pmatrix} 1 \\ -1 \\ z \end{pmatrix}, & \text{for } j = 0 \\ 0, & \text{for } j \geqslant 1 \end{cases} \tag{5.11}$$

and for all i, j's

$$y(i,j) = \overline{C}x\,(i,j) = 0.$$

Hence the cone (4.12) cannot be observed, moreover relation (5.11) shows that this cone is an eigencone.

A consequence of Example 5.3 is that the system considered, being not modally controllable, is not minimal as we had properly pointed out in Example 4.2. However this raises two important questions.

(i) The existence of a reduction procedure for nonobservable (noncontrollable) systems. In Part I we gave an algorithm to extract the GCRD of two 2-D polynomial matrices so that if

$$\begin{pmatrix} zI_n & \\ & \omega I_m \end{pmatrix} - A$$

and C are not right coprime we can exhibit $R(z,\omega)$, their common factor and

$$\begin{pmatrix} zI_n & 0 \\ 0 & \omega I_m \end{pmatrix} - A = \overline{A}R$$
$$C = \overline{C}R.$$

However, $\overline{A}$ and $\overline{C}$ are not any longer in state space form. Yet we know that $\delta_z|\overline{A}| < n$ or $\delta_\omega|\overline{A}| < m$, so that our ability to exhibit a state-space model of lower dimension depends on the existence of procedures to transform $\overline{A}$ in a form which would be row (or column) proper in both z and ω.

(ii) The existence of minimal realizations, i.e., of order $n+m$ for irreducible transfer functions $H(z,\omega) = b(z,\omega)/a(z,\omega)$ such that $\delta_z a = n$, $\delta_\omega a = m$. Clearly (i) and (ii) are very related questions, which we shall discuss in Section VI.

VI. Minimality of State-Space Model

It is shown in the last section that only a state-space realization with order (n, m)—i.e., the same order as the transfer function—can be both modally controllable and modally observable. Now the question is whether such a realization exists at all.

The best way to prove the existence of such realization is by construction. Note that, in the 2-D state-space model, the particular transform

$$\begin{bmatrix} \tilde{x}_h \\ \tilde{x}_v \end{bmatrix} = \begin{bmatrix} T_h & 0 \\ 0 & T_v \end{bmatrix} \begin{bmatrix} x_h \\ x_v \end{bmatrix} = T \begin{bmatrix} x_h \\ x_v \end{bmatrix} \tag{6.1}$$

enables us to change the basis of the state-space. The matrices $\{A, B, C, D\}$ are transformed to

$$\tilde{A} = TAT^{-1} \qquad \tilde{B} = TB$$
$$\tilde{C} = CT^{-1} \qquad \tilde{D} = D. \tag{6.2}$$

In fact, it is more convenient to work with a canonical form under the "similarity transform" defined by (6.2).

In the 1-D case, all minimal state-space model can be transformed to the controller canonical form. Similarly, almost all[1] 2-D state-space model can be transformed to the following *modal controller form* $\{A, B, C,\}$ (assuming $D = 0$)

$$A = \left[\begin{array}{ccccc|ccccc} -a_{10} & \cdots & \cdots & \cdots & -a_{n0} & & & & & \\ 1 & & & & & & & & & \\ & 1 & & \bigcirc & & & & A_{12} & & \\ & & \ddots & & & & & & & \\ \bigcirc & & & \ddots & & & & & & \\ & & & 1 & 0 & & & & & \\ \hline & & & & & -a_{01} & \cdots & \cdots & \cdots & -a_{0m} \\ & & & & & 1 & & & \bigcirc & \\ & & A_{21} & & & & 1 & & & \\ & & & & & & & \ddots & & \\ & & & & & \bigcirc & & & \ddots & \\ & & & & & & & & 1 & 0 \end{array}\right] \qquad B = \begin{array}{c} \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n \\ \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} m \end{array}$$

$$C = [b_{10} \cdots\cdots b_{n0} \mid b_{01} \cdots\cdots b_{0m}]. \tag{6.3}$$

where the entries of A_{12} and A_{21} are to be chosen such that

$$\det\left[\begin{pmatrix} zI_n & \\ & \omega I_m \end{pmatrix} - A\right] = a(z,\omega) \tag{6.4}$$

and

$$\det\left[\begin{array}{c|c} \begin{pmatrix} zI_n & \\ & \omega I_m \end{pmatrix} - A & B \\ \hline -C & 0 \end{array}\right] = b(z,\omega). \tag{6.5}$$

It is easy to check that, in (6.4), the coefficients $\{a_{i0}, 0 \leqslant i \leqslant n\}$ and $\{a_{0j}, 0 \leqslant j \leqslant m\}$ have already been matched. Similarly, in (6.4), the coefficients $\{b_{i0}, 0 \leqslant i \leqslant n\}$ and $\{b_{0j}, 0 \leqslant j \leqslant m\}$ have also been matched. Therefore only $2nm$ coefficients $\{a_{ij}, 1 < i \leqslant n, 1 \leqslant j \leqslant m\}$ and $\{b_{ij}, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ are to be matched. In other words, there are totally $2nm$ (nonlinear) equations to be satisfied. Coincidently, the number of free parameters in matrices A_{12} and A_{21} is also $2nm$. Therefore it is natural to conjecture that a solution (or, more precisely, a finite number of solutions) should always exist.

Now let us examine the plausibility of this conjecture by taking a low-order example.[2]

Example 6.1: (1, 1) Order Case—For ease of notation, let $A_{12} = \alpha$, $A_{21} = \beta$. Also (without loss of generality) let us assume that $b_{10} \neq 0$ (otherwise, we may have to use another canonical form). Then (4) becomes

$$z\omega + a_{01}z + a_{10}\omega - \alpha\beta = z\omega + a_{01}z + a_{10}\omega + a_{11}$$

or equivalently

$$\alpha\beta = -a_{11} \tag{6.6}$$

and (6.5) becomes

$$b_{01}z + b_{10}\omega + (a_{10}b_{01} + a_{01}b_{10} + b_{10}\alpha + b_{01}\beta)$$
$$= b_{01}z + b_{10}\omega + b_{11}$$

or

$$b_{01}\beta + b_{10}\alpha = b_{11} - a_{01}b_{10} - a_{10}b_{01}. \tag{6.7}$$

[1] Unlike 1-D case, there are some 2-D state-space systems $\{A, B, C, D\}$ which cannot be transformed into this particular canonical model. However, some modified canonical model can always be used as a replacement.

On the understanding that some modification on canonical form may be necessary, here we shall concentrate on the "modal controller form."

[2] E. Sontag (University of Florida, Gainesville, FL) independently arrived at the same conjecture (private communication).

Since $b_{10} \neq 0$, (6.6) and (6.7) have solutions

$$\alpha = \frac{1}{2b_{10}}\left(b_{10} - a_{01}b_{10} - a_{10}b_{01} \pm \sqrt{(b_{11} - a_{10}b_{10} - a_{10}b_{01})^2 - 4a_{11}b_{01}b_{10}}\right)$$
$$\beta = -a_{11}/\alpha. \tag{6.8}$$

Therefore, the existence of the (1, 1) order state-space model has been proved by construction. ∎

Unfortunately, (6.4) and (6.5) usually give a set of $2nm$ *nonlinear* equations; therefore, the solution may not always be in real numbers. For realization with real-gain constraints, we often need a realization order higher than $n + m$. To show that an (n, m) order real-gain realization may not exist, it is easiest to work on an example.

Example 6.2: The problem is to show that there is no (1, 1) order real-gain realization for the transfer function $(z + \omega)/(z\omega - 1)$.

Solution: Let us assume

$$A = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} e \\ f \end{bmatrix}$$
$$C = [g, h]. \tag{6.9}$$

Since $a_{11} = -1$, $\beta = \alpha^{-1}$. Then (6.4) is satisfied, and (6.5) becomes

$$fhz + eg\omega - (eh\alpha^{-1} + gf\alpha) = z + \omega \tag{6.10}$$

or equivalently,

$$fh = 1 \tag{6.11}$$
$$eg = 1 \tag{6.12}$$
$$eh\alpha^{-1} + gf\alpha = 0. \tag{6.13}$$

Now, (6.13) X hg − (6.11) X $g^2\alpha$ − (6.12) X $h^2\alpha^{-1}$ gives

$$g^2\alpha + h^2\alpha^{-1} = 0. \tag{6.14}$$

Since (6.14) has no real number solution, no (1, 1) order real-gain realization exists. ∎

In the practical aspect, real-gain realizations are much more desirable than complex realizations because the former are much easier to physically implement. Therefore, our $(2m + n)$ order real-gain realization (cf. Section III) are justified to be *practical* and *low-order* realizations. Indeed, for the transfer function in Example 6.2, the *minimal real-gain* realization $\{A, B, C\}$ can be obtained by our realization method

$$A = \left[\begin{array}{c|cc} 0 & -1 & 0 \\ \hline 1 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right]$$
$$B = \left[\begin{array}{c} 1 \\ \hline 0 \\ 1 \end{array}\right]$$
$$C = [1 \quad 0 \quad 1].$$

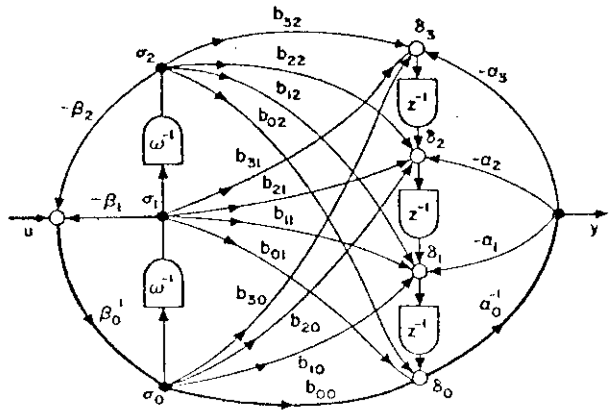


Fig. 7.

A. Special Transfer Functions

In designing digital filter, the transfer function may be intendedly chosen in a certain form for the purpose of an easier and/or better realization. Therefore, it is worth mentioning that some special types of transfer function can be easily realized in $(n + m)$ order real-gain realizations. There are two important special types of transfer functions:

(i) with separable denominator;
(ii) with separable numerator.

Let us first consider the separable denominator case. Assuming

$$H(z^{-1}, \omega^{-1}) = \frac{b(z^{-1}, \omega^{-1})}{a(z^{-1}, \omega^{-1})} = \frac{b(z^{-1}, \omega^{-1})}{\alpha(z^{-1})\beta(\omega^{-1})}$$
$$= \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}z^{-1}\omega^{-j}}{(\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_n z^{-n})(\beta_0 + \beta_1^{-1} + \cdots + \beta_m\omega^{-m})},$$
$$\alpha_0 \neq 0, \quad \beta_0 \neq 0 \tag{6.16}$$

then its circuit realization is shown in Fig. 7.

B. Special Form I

Let us check the validity of this realization. Note that in Fig. 7 the left-most "core" circuit is in the "controller form" so the transfer function between the input node and node σ_j (cf. Fig. 7) is

$$g_{\mathrm{in},\sigma_j} = \frac{\omega^{-j}}{\beta(\omega^{-1})}, \quad j = 1, \cdots, m.$$

Also note that in Fig. 7, the right-most part is in the "observer form" so the transfer function between node δ_i and the output node is

$$g_{\delta_i,\mathrm{out}} = \frac{z^{-i}}{\alpha(z^{-1})}, \quad i = 1, \cdots, n.$$

Therefore, the overall transfer function is

$$\sum b_{ij} \frac{z^{-i}}{\alpha(z^{-1})} \frac{\omega^{-j}}{\beta(\omega^{-1})}$$

which is clearly equal to $H(z^{-1}, \omega^{-1})$ of (6.16).

Secondly, let us consider the *separable numerator* case, which is to say a system with transfer function

$$\tilde{H}(z^{-1}, \omega^{-1}) = [H(z^{-1}, \omega^{-1})]^{-1} = \frac{\alpha(z^{-1})\beta(\omega^{-1})}{\sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}z^{-i}\omega^{-j}} \tag{6.17}$$

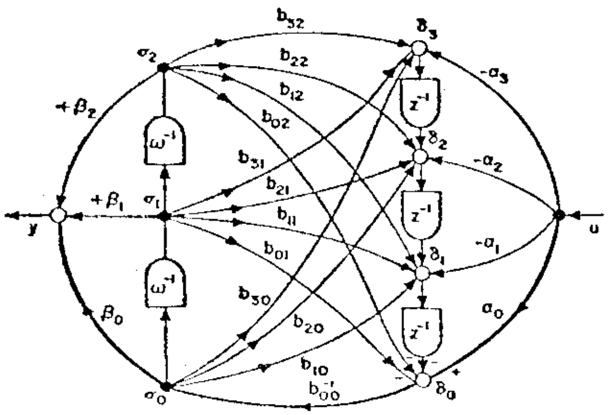


Fig. 8.

At first sight, it seems quite difficult. However, in actuality, the realization can be readily obtained by using the inversion rule by Kung [17]. More precisely, to realize the inverse system of Fig. 7, we first note that the path "input $\cdots \sigma_0 \cdots \delta_0 \cdots$ output" is a "feedthrough" path (i.e., a path connecting input and output with only constant gains). The second step is to invert all the gains and reverse all the arrows on the path (in our case, replace b_{00} by b_{00}^{-1}). Lastly, change signs of the gains of the branches which are entering this path. These steps complete the realization of $H^{-1}(z^{-1}, \omega^{-1})$ as shown in Fig. 8.

C. Special Form II

Finally, we would like to point out that, in the usual design problem, the constraint on numerator is much weaker than on denominator, hence the *special form II* seems to have higher potential in practical applications.

Appendix A1
The Algebraic Realization of 2-D State-Space Models

The algebraic setting is the same as in Fornasini-Marchesini [3], [4] or Wyman [18]. For convenience, we summarize some of the notations.

A. The External Representation for 2-D Filters

We shall consider a discrete, linear, causal, 2-D invariant, single-input–single-output (SISO) 2-D filter. That is to say

$T = \mathbb{Z} \times \mathbb{Z}$ is the *time set*, $\mathbb{Z} \times \mathbb{Z}$ has the partial order

$$(i, j) \leqslant (l, m) \text{ iff } i \leqslant l \quad \text{and} \quad j \leqslant m.$$

The past of the point (i, j) is $P(i, j) = \{(l, m) : (l, m) \leqslant (i, j)\}$ and its future is $F(i, j) = \{(l, m) : (i, j) \leqslant (l, m)\}$.

Since we consider a SISO filter, U (the set of inputs) $\equiv Y$ (the set of outputs) $= K$, an arbitrary field.

Now, $K^{\mathbb{Z}\times\mathbb{Z}}$ is the vector space of sequences $\{s_{ij}, i \in \mathbb{Z}, j \in \mathbb{Z}, s_{ij} \in K\}$ and if $s \in K^{\mathbb{Z}\times\mathbb{Z}}$ the support of s is

$$\operatorname{supp} s = \{(i, j) \in \mathbb{Z}\times\mathbb{Z} : s_{ij} \neq 0\}.$$

$s \in K^{\mathbb{Z}\times\mathbb{Z}}$ is said to be of *past finite support* (see Mullans and Elliott [7]) if $\operatorname{supp} s \cap P(i, j)$ is finite for all $(i, j) \in \mathbb{Z}\times\mathbb{Z}$.

$\mathcal{P}$, the set of past finite sequences is a subvector space of $K^{\mathbb{Z}\times\mathbb{Z}}$ and we take $\mathcal{U}$ (the strings of inputs) $\equiv \mathcal{Y}$ (the strings of outputs) $\triangleq \mathcal{P}$.[3] z and ω, the shifts on $\mathcal{P}$, are defined by $z(s)_{ij} = s_{i+1,j}$ and $\omega(s)_{ij} = s_{i,j+1}$ so that $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ can be represented by

$$u(z, \omega) \triangleq \sum_{h,k} u_{hk} z^{-h} \omega^{-k} \quad \text{and} \quad y(z, \omega) \triangleq \sum_{h,k} y_{hk} z^{-h} \omega^{-k}.$$

Then, the 2-D filter is given by the *2-D input–output map* $\mathcal{F}: \mathcal{U} \to \mathcal{Y}$. $\mathcal{F}$ is assumed linear, causal, and 2-D invariant.

By *2-D invariant*, we mean that $\mathcal{F}$ commutes with the shifts z and ω

$$\mathcal{F}z = z\mathcal{F} \quad \text{and} \quad \mathcal{F}\omega = \omega\mathcal{F}.$$

By *causal*, we mean that the spread function of an impulse at time $(0, 0)$ is contained in $F_{(0,0)}$ the future of $(0, 0)$.

Hence $F(z, \omega)$, the *2-D transfer function* corresponding to $\mathcal{F}$, is

$$F(z, \omega) = \sum_{i \geqslant 0, j \geqslant 0} f_{ij} z^{-i} \omega^{-j} \in K[[z^{-1}, \omega^{-1}]] \tag{A1.1}$$

and

$$y(z, \omega) = F(z, \omega) u(z, \omega)$$

is the (z, ω) transform input–output relation.

In the following, we shall consider that

$$F(z, \omega) = \frac{b(z, \omega)}{a(z, \omega)}$$

is rational. In order to ensure causality, we assume that the polynomial a is monic

$$a(z, \omega) = z^n \omega^m + \sum_{(0,0) < (i,j) \leqslant (n,m)} a_{ij} z^{n-i} \omega^{m-j} \tag{A1.2}$$

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\mathcal{F}"] & \mathcal{P} \arrow[d, "\pi_F"] \\
\mathcal{P}_P \arrow[u, "i_P"] \arrow[r, "\bar{\mathcal{F}}"] & \mathcal{P}_F
\end{tikzcd}$$

Fig. 9.

B. Nerode Equivalence

Clearly, $\mathcal{F}$ has a $K[z, \omega]$ module structure and since $\mathcal{F}$ is a 2-D invariant, it is a $K[z, \omega]$ module homomorphism. Now as in the 1-D case (see [15]) we decompose $\mathcal{P}$ as $\mathcal{P}_P \oplus \mathcal{P}_F$ where $\mathcal{P}_P$ is the space of past sequences and $\mathcal{P}_F$ the space of future sequences.

The restricted input-output map $\bar{\mathcal{F}}$ is defined by Fig. 9, where i_P is the canonical injection of $\mathcal{P}_P$ in $\mathcal{P}$, and π_F the projection of $\mathcal{P}$ on $\mathcal{P}_F$. If $\mathcal{P}_P$ is (z, ω) invariant, $\bar{\mathcal{F}}$ is also a $K[z, \omega]$ module homomorphism, the shifts on $\mathcal{P}_P$ being just z and ω, and the shifts on $\mathcal{P}_F$ being defined by $z_F = \pi_F z$ and $\omega_F = \pi_F \omega$.

Remark: The decomposition of $\mathcal{P}$ as $\mathcal{P}_p \oplus \mathcal{P}_F$ where $\mathcal{P}_p$ is (z, ω) invariant corresponds to partitioning $\mathbb{Z} \times \mathbb{Z}$ into a "global past" and a "global future" with a cross cut of $\mathbb{Z} \times \mathbb{Z}$.

A *cross cut* $(\mathcal{C})$ of $\mathbb{Z} \times \mathbb{Z}$ (see Mullans and Elliott [7]) is a subset of $\mathbb{Z} \times \mathbb{Z}$ such that for all $(h, k) \in \mathbb{Z} \times \mathbb{Z}$ one and only one of the following is true: $(h, k) \in (\mathcal{C})$; $(i, j) < (h, k)$ for some $(i, j) \in (\mathcal{C})$, $(h, k) < (i, j)$ for some $(i, j) \in (\mathcal{C})$.

[3] The finite past condition ensures that for the output at time (i, j) $y_{ij} = \sum_{\operatorname{supp}(u) \cap P(i,j)} f_{i-h,j-k} u_{hk}$, the sum is finite.

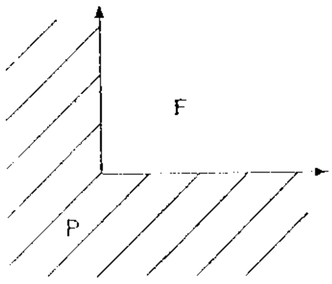


Fig. 10

$$\begin{tikzcd} \mathcal{P}_P \arrow[rr, "\overline{\mathcal{F}}"] \arrow[dr, "\overline{\mathcal{G}}"'] & & \mathcal{P}_F \\ & \overline{\mathcal{X}} \arrow[ur, "\overline{\mathcal{H}}"'] & \end{tikzcd}$$

Fig. 11.

Thus $(\mathcal{C})$ defines a partition of $\mathbb{Z}\times\mathbb{Z} : \mathbb{Z}\times\mathbb{Z} = P \cup F$

P (the global past)

$$= \{(h, k) : (h, k) \leqslant (i, j) \text{ for some } (i, j) \in (\mathcal{C})\}$$

F (the global future)

$$= \{(h, k) : (i, j) < (h, k) \text{ for some } (i, j) \in (\mathcal{C})\}.$$

The conditions imposed on the cross cuts guarantee that P is (z, ω) invariant (see Fig. 10). From now on, we shall consider the following cut of $\mathbb{Z}\times\mathbb{Z}$. This corresponds to $\mathcal{P}_F = z^{-1}\omega^{-1}K[[z^{-1}, \omega^{-1}]]$ and

$$\mathcal{P}_P = \mathcal{P}/\mathcal{P}_F \simeq K[z, \omega] \oplus \omega^{-1}(K[z][[\omega^{-1}]]) \oplus z^{-1}(K[\omega][[z^{-1}]]) \quad \text{(A.1.3)}$$

Note:

$$K[z]\,[[\omega^{-1}]] \neq K[[\omega^{-1}]]\,[z].$$

Now, as in the 1-D case, canonical state-space realizations are induced by the canonical factorizations of $\overline{\mathcal{F}}$, shown in Fig. 11, where $\overline{\mathcal{X}}$ is a $K[z, \omega]$ module, $\overline{\mathcal{G}}$ and $\overline{\mathcal{H}}$ are $K[z, \omega]$ module homomorphisms, $\overline{\mathcal{G}}$ is onto, $\overline{\mathcal{H}}$ is into.

Two particular canonical state-spaces of interest are $\overline{\mathcal{X}}_{co} \triangleq \mathcal{P}_P/\mathrm{Ker}\,\overline{\mathcal{F}}$ and $\overline{\mathcal{X}}_{ob} \triangleq \mathcal{R}(\overline{\mathcal{F}})$.

But since we have assumed that $F(z, \omega) = b(z, \omega)/a(z, \omega)$ is rational

$$\mathrm{Ker}\,\overline{\mathcal{F}} = a(z, \omega)\,\mathcal{P}_P \quad \text{and} \quad \mathcal{R}(\overline{\mathcal{F}}) = \{y \in \mathcal{P}_F : \pi_F\, a(z, \omega)\, y = 0\}. \quad \text{(A.1.4)}$$

Now, we shall study more closely the canonical realizations associated with $\overline{\mathcal{X}}_{co}$ and $\overline{\mathcal{X}}_{ob}$.

C. The Controllability and Observability Realizations: The Notions of Global and Local State-Space

We need first to study the structure of $\mathcal{P}_{P/a(z,\omega)\mathcal{P}_P}$. From (A.1.3) if $u \in \mathcal{P}_P$

$$u = \tilde{u}(z, \omega) + \sum_1^\infty u_i^\omega(z)\,\omega^{-i} + \sum_1^\infty u_j^z(\omega)\,z^{-j}$$

where

$$\tilde{u} \in K[z, \omega], \quad u_i^\omega \in K[z], \quad u_j^z \in K[\omega].$$

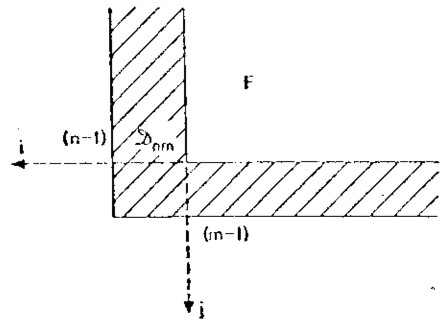


Fig. 12.

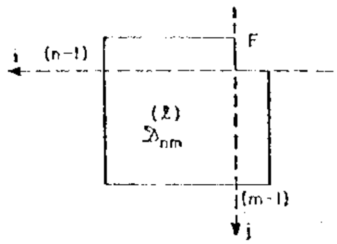


Fig. 13.

Now, in order to reduce $u \in \mathcal{P}_P$ modulo $a(z, \omega)\,\mathcal{P}_P$, we use the Euclidean division algorithm over $K(\omega)[z]$ (respectively $K(z)[\omega]$) i.e., if $\theta \in K(\omega)[z]$, there exists unique q and r in $K(\omega)[z]$ such that

$$\theta = qa + r, \quad \text{with} \quad \delta_z r < n = \delta_z a.$$

With this procedure $u(z, \omega) \in \mathcal{P}_P$ can be reduced to $x_{co}(z, \omega)$ such that

$$x_{co}(z, \omega) = \sum_{\mathfrak{D}_{nm}} x_{ij} z^i \omega^j \qquad \text{(A.1.5)}$$

where

$$\mathfrak{D}_{nm} = \{(i, j) : 0 \leqslant i \leqslant n-1, \ j \leqslant m-1 \ \text{ or } \ 0 \leqslant j \leqslant m-1, \ i \leqslant n-1\}$$

(see Fig. 12).

Hence the state-space is infinite dimensional, as has already been noted by Fornasini–Marchesini and Wyman. Fornasini and Marchesini (see [4]) failed to obtain the structure (A.1.5) for the state-space and consequently, as we remarked in Section IV, they introduced only a "partial" state-space model.

However, they made the interesting observation that while $\mathcal{H}_{co}(z, \omega)$ of (A.1.5) preserves all the past information, only a finite number of states are necessary to perform the recursions of the 2-D filter.

Indeed, since the 2-D system considered is causal, if we want to compute the output y at time $(1, 1)$, we need only a finite number of the states x_{ij} taken at time $(0, 0)$, namely those such that $(i, j) \in \mathfrak{D}_{nm}^{(l)}$, where

$$\mathfrak{D}_{nm}^{(l)} = \{(i, j) : (0, 0) \leqslant (i, j) \leqslant (n-1, m-1) \ \text{ or } i = -1, \ 0 \leqslant j \leqslant m-1 \text{ or } j = -1, \ 0 \leqslant i \leqslant n-1\}$$

(see Fig. 13).

Following the terminology of Fornasini and Marchesini, we shall call $\mathcal{X}_{co} = \{x_{ij} : (i, j) \in \mathfrak{D}_{nm}\}$ the *global state* associated with Nerode equivalence, while $\mathcal{X}_{co}^{(l)} = \{x_{ij} : (i, j) \in \mathfrak{D}_{nm}^{(l)}\}$ will be called the *local state*-space. The local state-space is of dimension $nm + n + m$ and can be decomposed into a *horizontal state* x_h, a *corner state* x_c and a *vertical state* x_v

$$x_h = \{x_{ij}: j = -1,\ 0 \leq i \leq n-1\} \text{ is of dimension } n$$
$$x_c = \{x_{ij}: (0,0) \leq (i,j) \leq (n-1, m-1)\} \text{ is of dimension } nm$$
$$x_v = \{x_{ij}: i = -1,\ 0 \leq j \leq m-1\} \text{ is of dimension } m.$$

The local state-space model associated with $\mathfrak{X}_{co}^{(l)}$ is obtained, as in the 1-D case, by studying the effect of the operations

$$zx_{co}(z,\omega) + u^z(\omega) \text{ modulo } a(z,\omega)\,\mathcal{P}_P \quad \text{where } u^z(\omega) = \sum_1^\infty u_{0j}\omega^{-j}$$

$$\omega x_{co}(z,\omega) + u^\omega(z) \text{ modulo } a(z,\omega)\,\mathcal{P}_P \quad \text{where } u^\omega(z) = \sum_1^\infty u_{i0} z^{-i} \qquad \text{(A.1.6)}$$

not only on the global space, but also on the local space $\mathfrak{X}_{co}(l)$.

The operations (A.1.6) correspond to shifting the past sequence of input either horizontally or vertically and concatenating some new inputs either along the vertical axis $i = 0$ or along the horizontal axis $j = 0$.

Let

$$a(z,\omega) = \sum_{i=0}^{n} a_i^z(\omega)\, z^{n-i} = \sum_{j=0}^{m} a_j^\omega(z)\, \omega^{m-j}$$

where

$$a_i^z(\omega) = \sum_{j=0}^{m} a_{ij}\omega^{m-j}, \quad a_j^\omega(z) = \sum_{i=0}^{n} a_{ij} z^{n-i}, \quad a_{00} = 1$$

(we have assumed that $a(z,\omega)$ is monic).

Now, let

$$H_i^z(\omega) = \frac{a_i^z(\omega)}{a_0^z(\omega)} = \sum_{l=0}^{\infty} h_l^{(i)z}\omega^{-l}$$

so that $h_0^{(i)z} = a_{i0}$ and also

$$H_j^\omega(z) = \frac{a_j^\omega(z)}{a_0^\omega(z)} = \sum_{s=0}^{\infty} h_s^{(j)\omega} z^{-s}$$

with $h_0^{(j)\omega} = a_{0j}$. Then in order to perform the reduction modulo $a(z,\omega)\,\mathcal{P}_P$ of the operations (A.1.6) we use the identities

$$z^n = -\sum_1^n \frac{a_i^z(\omega)}{a_0^z(\omega)} z^{n-i} = -\sum_1^n H_i^z(\omega)\, z^{n-i} \text{ modulo } a(z,\omega)\,\mathcal{P}_P \qquad \text{(A.1.7a)}$$

$$\omega^m = -\sum_1^m \frac{a_j^\omega(z)}{a_0^\omega(z)} \omega^{m-j} = -\sum_1^m H_j^\omega(z)\, \omega^{m-j} \text{ modulo } a(z,\omega)\,\mathcal{P}_P. \qquad \text{(A.1.7b)}$$

Then Fig. 14 describes the effect of the shifts z and ω on the local state.

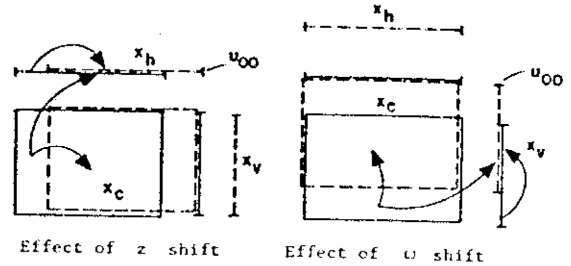


Fig. 14. Full line represents the old state and dotted line represents the new state. Arrows represent the reductions.

Thus the horizontal state x_h can only be propagated horizontally, the vertical state x_v can only be propagated vertically, and the corner state x_c can be propagated in both directions.

Now, if we index the states of Fig. 14 from right to left and top to bottom and if x_c^i is the ith row of x_c, we obtain the following state space model which is in 2-D *controllability form*

$$\begin{bmatrix} x_h(i+1,j) \\ \hline x_c^1(i+1,j) \\ \vdots \\ x_c^m(i+1,j) \\ \hline x_v(i,j+1) \end{bmatrix} = \left[\begin{array}{c|cccc|c} A_{hh} & A_{hc}^1 & \cdots & & A_{hc}^m & 0 \\ \hline & A_{hh} & A_{hc} & & A & S_1 \\ 0 & & \ddots & \ddots & & \vdots \\ & & & & A_{hc} & \vdots \\ & & & & A_{hh} & S_m \\ \hline 0 & & & 0 & A_{vc} & A_{vv} \end{array}\right] \overbrace{\begin{bmatrix} x_h(i,j) \\ \hline x_c^1(i,j) \\ \vdots \\ x_c^m(i,j) \\ \hline x_v(i,j) \end{bmatrix}}^{x(i,j)} + \begin{bmatrix} B_h^{co} \\ \hline 0 \\ \hline B_v^{co} \end{bmatrix} u(i,j) \qquad \text{(A1.8)}$$

$$y(i,j) = [C_h \mid C_c^1 \ \cdots \ C_c^m \mid C_v]\, x(i,j)$$

where

$$A_{hh} = \overbrace{\begin{bmatrix} 0 & & -h_0^{(n)z} \\ 1 & \bigcirc & \vdots \\ \bigcirc & 1 & -h_0^{(1)z} \end{bmatrix}}^{n} \Big\} n$$

$$A_{vv} = \overbrace{\begin{bmatrix} 0 & & -h_0^{(m)\omega} \\ 1 & \bigcirc & \vdots \\ \bigcirc & 1 & -h_0^{(1)\omega} \end{bmatrix}}^{m} \Big\} m$$

$$A_{hc}^{(l)} = \begin{array}{c} \overbrace{\hphantom{\bigcirc \quad -h_l^{(n)z}}}^{n} \\ \left[\begin{array}{cc} & -h_l^{(n)z} \\ \bigcirc & \vdots \\ & -h_l^{(1)z} \end{array}\right] \end{array} \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} n \qquad 1 \leqslant l \leqslant n$$

$$A_{vc}^{m} = \begin{array}{c} \overbrace{\hphantom{-h_1^{(m)\omega} \cdots\cdots -h_n^{(m)\omega}}}^{n} \\ \left[\begin{array}{ccc} -h_1^{(m)\omega} & \cdots\cdots & -h_n^{(m)\omega} \\ \vdots & & \vdots \\ -h_1^{(1)\omega} & \cdots\cdots & -h_n^{(1)\omega} \end{array}\right] \end{array} \left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} m$$

$$S_l = \begin{array}{c} \\ l \end{array} \begin{array}{c} \hphantom{---} l \\ \left[\begin{array}{ccc} & \vdots & \\ \cdots & 1 & \\ & & \end{array}\right] \end{array} \qquad 1 \leqslant l \leqslant m$$

$$B_h^{c0} = \left.\begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}\right\} n \qquad B_v^{c0} = \left.\begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}\right\} m$$

$$C_h = [h_{10} \cdots\cdots h_{n0}] \quad C_v = [h_{01} \cdots\cdots h_{0m}]$$

$$C_c^l = [h_{1l} \cdots\cdots h_{nl}].$$

The h_{ij}'s are the Markov parameters of the 2-D filter, i.e.,

$$\frac{b(z,\omega)}{a(z,\omega)} = \sum_{i,j} h_{ij} z^{-1} \omega^{-j}.$$

The state-space model (A1.8) is clearly in Roesser's form. However, we could have propagated the corner state vertically instead of horizontally.

Remark: This suggests another type of model which would have three types of states: horizontal of size n, vertical of size m, corner of size nm. These states could be propagated respectively horizontally, vertically, and in both directions. Physically such a model would correspond to a grid constituted of delay elements z^{-1} and ω^{-1}. However these models are not minimal from the point of view of Roesser's model and the number of states: $n + m + nm$ is very large.

The *2-D observability form* can be derived in a similar way, indeed $\mathfrak{X}_{ob}$ is the space of the

$$y(z,\omega) = \sum_{(1,1) \leqslant (i,j)} y_{ij} z^{-i} \omega^{-j}$$

such that $\pi_F a(z,\omega) y(z,\omega) = 0$. But these y's can be parameterized by the y_{ij}'s such that $1 \leqslant i \leqslant n$, $j \geqslant 1$ or $i \geqslant 1$, $1 \leqslant j \leqslant m$. Indeed, since the coefficient of $z^{-1}\omega^{-1}$ in $a(z,\omega) y(z,\omega)$ is 0

$$y_{n+1,m+1} = - \sum_{(0,0)<(i,j)\leqslant(n,m)} a_{ij} y_{n+1-i,m+1-j} \quad \text{(A1.9)}$$

similarly all the y_{ij}'s such that $i \geqslant n+1$ and $j \geqslant m+1$ can be computed.

As a consequence $\mathfrak{X}_{ob} = \{y_{ij} : 1 \leqslant i \leqslant n,\ j \geqslant 1 \text{ or } i \geqslant 1,\ 1 \leqslant j \leqslant m\}$ is the observability global space. Also, since $y_{n+1,m+1}$ depends only on the y_{ij}'s such that $(1,1) \leqslant (i,j) < (n+1, m+1)$, the local space is $\mathfrak{X}_{ob}^{(l)} = \{y_{ij} : (1,1) \leqslant (i,j) < (n+1, m+1)\}$ and is of dimension $nm + n + m$. This local state can also be decomposed into a horizontal state of dimension n, a vertical state of dimension m and a corner state of dimension nm.

The corresponding matrices (A, B, C) can be obtained easily. Also, some other canonical forms can be derived: for example, the controller form arises by factoring

$$\overline{F} = \pi_F\, b(z,\omega) \times \pi_F \frac{1}{a(z,\omega)}$$

and by realizing

$$\pi_F \frac{1}{a(z,\omega)}$$

in observability form and then composing with $\pi_F b(z,\omega)$.

Moreover, although the previous models are of size $nm + n + m$ and hence are not minimal from the point of view of Roesser's model, their size can be reduced easily. A special reduction procedure on the controller form yields the implementation of size $n + 2m$ obtained in Section III.

Appendix A2
Some Notations for the 2-D Z-Transform

We consider the sequence $x(i,j)$, $i \geqslant 0$, $j \geqslant 0$. Then

$$Z\,\{x(i,j)\} = x(z,\omega) \triangleq \sum_{i\geqslant 0\ j\geqslant 0} x(i,j)\, z^{-i}\omega^{-j}$$

and

$$Z\,\{x(i+1,j)\} = \sum_{i\geqslant 0\ j\geqslant 0} x(i+1,j)\, z^{-i}\omega^{-j}$$

$$= z(x(z,\omega) - x_{\delta_v}(\omega)) \quad \text{(A2.1)}$$

where

$$x_{\delta_v}(\omega) = \sum_{j\geqslant 0} x(0,j)\,\omega^{-j}$$

$$\delta_v = \{(0,j),\ j \in \eta\} \text{ is the vertical axis.}$$

Similarly

$$Z\,\{x(i,j+1)\} = \omega(x(z,\omega) - x\delta_h(z)) \quad \text{(A2.2)}$$

where

$$x_{\delta_h}(z) = \sum_{i\geqslant 0} x(i,0)\, z^{-i}$$

and $\delta_h = \{(i,0),\ i \in \eta\}$ is the horizontal axis.

References

[1] S. Attasi, "Systèmes lineaires homogènes à deux indices," *Rapport Laboria*, no. 31, Sept. 1973.
[2] ——, "Modélisation et traitement des suites à deux indices," *Rapport Laboria*, Sept. 1975.
[3] E. Fornasini and G. Marchesini, "Algebraic realization theory of two-dimensional filters," in *Variable Structure Systems*, A. Ruberti and R. Mohler, Eds. New York: Springer Verlag (Springer Lecture Notes in Economics and Mathematical Systems), 1975.
[4] ——, "State-space realization theory of two-dimensional filters," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 484–492, Aug. 1976.
[5] D. D. Givone and R. P. Roesser, "Minimization of multidimensional linear iterative circuits," *IEEE Trans. Comput.*, vol. C-22, pp. 673–678, July 1973.
[6] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer-Verlag, 1974.

[7] R. E. Mullans and D. L. Elliott, "Linear systems on partially ordered time sets," in *Proc. 1973 IEEE Conf. Decision and Control*, pp. 334–337, 1973.
[8] D. D. Givone and R. P. Roesser, "Multidimensional linear iterative circuits—general properties," *IEEE Trans. Comput.*, vol. C-21, pp. 1067–1073, Oct. 1972.
[9] S. K. Mitra, A. D. Sagar, and N. A. Pendergrass, "Realizations of two-dimensional recursive digital filters," *IEEE Trans. Circuits and Systems*, vol. CAS-22, pp. 177–184, Mar. 1975.
[10] V. Zakian and N. S. Williams, "Canonical state equations of delay differential systems," Control Systems Centre, Rep. 213, Univ. Manchester, Manchester, England, July 1973.
[11] C. T. Chen, *Introduction to Linear System Theory*. New York: Holt, Rinehart and Winston, 1970.
[12] R. R. Read, J. L. Shanks, and S. Treitel, "Two-dimensional recursive filtering," in *Topics in Applied Physics*. New York: Springer-Verlag, 1975, vol. 6, pp. 137–176.
[13] R. P. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 1–10, Feb. 1975.
[14] B. Vilfan, "Another proof of the two-dimensional Cayley–Hamilton theorem," *IEEE Trans. Comput.*, vol. C-22, p. 1140, Dec. 1973.
[15] R. E. Kalman, P. Falb, and M. Arbib, *Topics in Mathematical System Theory*. New York: McGraw-Hill (International Series in Pure and Applied Mathematics), 1969.
[16] H. H. Rosenbrock, *State Space and Multivariable Theory*. New York: Wiley, 1970.
[17] S. Y. Kung, "An easy method for inverse system realizations and its applications," to be submitted.