General Logic
(with Special Application to Relevance Logic)

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1 Structural Rules

Our formulation of a natural deduction system for general logic is due to Read [Rea88, pp. 51–71] and Slaney [Sla90].

Definition 1.1. The collection of *dependencies* is defined inductively.

1. The following are dependencies: T, t.
2. The following are dependencies: $0, 1, 2, 3, 4, \dots$
3. If x and y are dependencies then (x, y) is a dependency.
4. If x and y are dependencies then $(x; y)$ is a dependency.

As usual, we allow ourselves to drop the outermost parentheses. So, the following is a dependency:

$$7; ((4; 3), 2)$$

The following is also a dependency:

$$(7; (t, 7)), T$$

Henceforth, we never allow empty dependency sets (as is permitted in Lemmon's system [Lem78]). Instead, the dependencies t and T will play the role that is played by the empty set in the traditional formulation.[1] So, the line

$$t \quad (n)\ A$$

is what we need to get in order to show that A is derivable from no assumptions. We will explain shortly how one can obtain t as a dependency by itself, even though one always begins with 1 as the first dependency.

What is the meaning of the comma and semicolon? That's not so clear. For now, we use some suggestive terminology: comma corresponds to aggregation, and semicolon corresponds to fusion. Intuitively, a fusion of sentences does necessarily strengthen the sentences.

We now give some meaning to the two constants t and T. Intuitively speaking, t is the set of all tautologies, and T is the emptyset. Formally, the two basic rules for using T and t are:

t Identity

$$t; X = X$$

In natural deduction format:

$t; X$	$(i)\ A$	
X	$(j)\ A$	$i,\ t{=}$

T Identity

$$T, X = X$$

[1] Those who know some category theory will quickly realize what is going on here: T is a zero element the category of dependencies, and when ";" is associative, t is a monoidal unit.

In natural deduction format:

T, X	$(i)\ A$	
X	$(j)\ A$	$i,\ T=$

Given the structural rules for the comma (given below), we can always simply omit T. Just treat it as you are wont to treat the empty set!

	Comma	Semicolon	System
$X * (Y * Z) = (X * Y) * Z$	SR1	Associativity	
$X * Y = Y * X$	SR2	Commutativity	**RW**
$X * X \Leftarrow X$	SR3	Contraction	**R**
$X \Leftarrow X * X$	SR3	Mingle	**RM**
$X \Leftarrow X * Y$	SR4	Expansion	**CPL**

The top two rows (associativity, commutativity) are two-sided replacement rules. That means, that given either side of an equality as a dependency, we can replace it on a subsequent line with the other side of that equality. For example, SR2 tells us that we can replace X, Y with Y, X.

Postulate: We assume all structural rules for the comma SR1–SR5. Hence, we are permitted to make the following moves in a proof.

1	$(1)\ A$	A
$1, T$	$(2)\ A$	$1, T =$
$T, 1$	$(3)\ A$	$2,$ SR2
$(T, 1), (8, 1)$	$(4)\ A$	$3,$ SR4
$T, (1, (8, 1))$	$(5)\ A$	$4,$ SR1

The comma rules follow automatically for Lemmon's system, because it is implicitly assumed that dependencies aggregate by set union. (Verify for yourself that set union obeys SR1–SR3, and that SR4 is a consequence of the fact that we can use the conjunction trick to add arbitrary dependencies.) If you think about it, you will realize that this is definitely a non-trivial assumption that we made (without realizing it) about how deduction works. That's why we introduce the semicolon, because it's not completely obvious that all relevant operations on dependencies obey the same structural rules as the set theoretic operations.

2 Inference Rules

Rule of Assumptions (A)

$$i \quad (i)\ A \qquad \mathrm{A}$$

We apologize for the confusing notation: A (italicized) is a sentence, whereas A (roman) is a citation.

In words:

> On any line, you are permitted to write any sentence, so long as you write the number of the line as the dependency and cite the rule "A".

2.1 Copying Rules

Double Negation (DN)

$$\begin{array}{lll} X & (i)\ A & \\ X & (j)\ --A & i,\ \mathrm{DN} \end{array}$$

$$\begin{array}{lll} X & (i)\ --A & \\ X & (j)\ A & i,\ \mathrm{DN} \end{array}$$

∨ Introduction

$$\begin{array}{lll} X & (i)\ A & \\ X & (j)\ A \vee B & i,\ \vee\text{Intro} \end{array}$$

$$\begin{array}{lll} X & (i)\ A & \\ X & (j)\ B \vee A & i,\ \vee\text{Intro} \end{array}$$

∧ Elimination

$$\begin{array}{lll} X & (i)\ A \wedge B & \\ X & (j)\ A & i,\ \wedge\text{Elim} \end{array}$$

$$\begin{array}{lll} X & (i)\ A \wedge B & \\ X & (j)\ B & i,\ \wedge\text{Elim} \end{array}$$

2.2 Aggregation Rules

∧ Introduction

$$\begin{array}{lll} X & (i)\ A & \\ Y & (j)\ B & \\ X, Y & (k)\ A \wedge B & i, j,\ \wedge\text{Intro} \end{array}$$

2.3 Fusion Rules

We have to be careful now in the statement of these rules, because they depend crucially on the structure of the dependency set.

Conditional Proof (CP)

i	$(i)\ A$	A
$X; i$	$(j)\ B$	
X	$(k)\ A \to B$	$i, j,$ CP

In words:

> We can remove dependencies that are fusioned to the *right* by placing the corresponding sentence as the antecedent of a conditional.

In practice, the modified CP rules renders useless any making of irrelevant assumptions. Because, if we already have a derivation of B from X (i.e. a line with dependency X and B on the right), there is no guarantee that we will be able to get a derivation of B from the dependency $X; A$. Roughly speaking, the fusion of X and A might be logically weaker than X.

Note: CP only applies to dependencies that are attached to the right. For example, the following step is *not* warranted by CP:

i	$(i)\ A$
$i; X$	$(j)\ B$
X	$(k)\ A \to B$

However, when the semicolon is commutative, then one additional step will transform this into a valid inference.

Modus Ponendo Ponens (MPP)

X	$(i)\ A \to B$	
Y	$(j)\ A$	
$X; Y$	$(k)\ B$	i, j MPP

Modus Tolendo Tollens (MTT)

X	$(i)\ A \to B$	
Y	$(j)\ -B$	
$X;Y$	$(k)\ -A$	i,j MTT

$\vee$ Elimination

X	$(i)\ A \vee B$	
j	$(j)\ A$	A
$Y;j$	$(k)\ C$	
l	$(l)\ B$	A
$Z;l$	$(m)\ C$	
$X;(Y;Z)$	$(n)\ C$	i,j,k,l,m $\vee$ Elim

3 Some Systems of Relevance Logic

Definition 3.1. Consider some set of the structural and inference rules. Let $A_1, \ldots, A_n, B$ be sentences. Then we say that

$$A_1, \ldots, A_n \vdash B,$$

just in case there is a correctly written proof with final line

$$1, 2, \ldots, n \quad (m)\ B$$

where A_i is assumed on line i, for $i = 1, \ldots, n$.

Postulate: All systems obey all structural rules for comma.

Postulate: All systems have the full set of inference rules.

3.1 DW

DW is the system that assumes no structural rules for ";" other than t Identity. However, t Identity yields some tautologies.

We now show that $\vdash P \to P$ in DW.

1	(1) P	A
$t;1$	(2) P	$1, t =$
t	(3) $P \to P$	$1, 2$, CP

In fact, it can be shown that the only tautologies in DW are of the form $A \to A$.

3.2 Associative, Commutative Logic (RW)

The system **RW** assumes associativity and commutativity for semicolon.

Things that can be proven in RW:

<u>Permutation</u> $\vdash [P \to (Q \to R)] \to [Q \to (P \to R)]$

<u>Prefixing</u> $\vdash [P \to Q] \to [(R \to P) \to (R \to Q)]$

<u>Suffixing</u> $\vdash [P \to Q] \to [(Q \to R) \to (P \to R)]$

<u>Contrapositive</u> $\vdash (P \to Q) \to (-Q \to -P)$

3.2.1 Fusion

Definition 3.2. Consider a logic with connectives $\rightarrow$ and $\circ$. We say that $\circ$ is a *fusion* connective just in case the following holds for all formulas A, B, C:

$$A \rightarrow (B \rightarrow C) \quad \text{is logically equivalent to} \quad (A \circ B) \rightarrow C.$$

In logics with Associative and Commutative ";", we can show that the connective $\circ$ defined in terms of:

$$A \circ B = -(A \rightarrow -B),$$

is a fusion connective.

3.3 R

The system R (= RW + Contraction) is the most important "relevance logic"; it is the favorite of the pioneers, Anderson and Belnap [AB75]. It adds the structural rule *Contraction*:

$$X; X \Leftarrow X$$

to the structural rules of Associative, Commutative Logic. In natural deduction format:

$X; X$	$(i)\ A$	
X	$(j)\ A$	i, Contract

Things that can be proven in **R** but not in **RW**:

Contraction	$\vdash (P \to (P \to Q)) \to (P \to Q)$
Disjunctive Consequence	$P \to Q \vdash -P \vee Q$
Proof by Contradiction	$\vdash (P \to Q) \to [(P \to -Q) \to -P]$
Weak Reductio	$\vdash (P \to -P) \to -P$
Reductio	$\vdash (-P \to P) \to P$
Mixing	$\{P \to R, -Q \to R\} \vdash (Q \to P) \to R$
Dichotomy	$\vdash (-P \to Q) \to [(P \to Q) \to Q]$
Excluded Middle	$\vdash P \vee -P$

Conjecture: Suppose that A, B are formulas that do not contain "$\to$". Then $A \vdash B$ in **R** iff B is a first degree entailment (FDE) of A. If so, this means that **R** is a conservative extension of FDE.

This conjecture is false.

3.4 RM

The system **RM** (= R + Mingle) is slightly irrelevant. Its structural rule is *Mingle*

$$X \Leftarrow X; X,$$

which is the reverse of Contraction. The natural deduction format of Mingle is stated below.

X	$(i)\ A$	
$X; X$	$(j)\ A$	i, Mingle

Things that can be proven in **RM** but not in **R**:

Mingle	$\vdash P \to (P \to P)$
Chain	$\vdash (P \to Q) \vee (Q \to P)$

Here's a proof of the sentence Mingle using the structural rule Mingle.

1	$(1)\ P$	A
$1;1$	$(2)\ P$	1, Mingle
$t;(1;1)$	$(3)\ P$	$2, t =$
$(t;1);1$	$(4)\ P$	3, Assoc
$t;1$	$(5)\ P \to P$	$1,4$, CP
t	$(6)\ P \to (P \to P)$	$1,5$, CP

3.4.1 Semantics for RM

The natural deduction system **RM** is sound and complete relative to Sugihara's semantics, as described below.

Definition 3.3. A *Sugihara valuation* f is a function from sentences to integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ such that:

1. $f(A \wedge B) = \min\{f(A), f(B)\}$.
2. $f(A \vee B) = \max\{f(A), f(B)\}$.
3. $f(-A) = -f(A)$. (The "$-$" serves double duty here as the negation symbol in our language and as the subtraction operator on numbers.)
4. $f(A \to B) = \text{sign}[f(B) - f(A)] \times \max\{|f(A)|, |f(B)|\}$, where $\text{sign}(z) = +1$ when $z \geq 0$, and $\text{sign}(z) = -1$ when $z < 0$.

Definition 3.4 (Designated Values). We say that a Sugihara valuation f satisfies a sentence A just in case $f(A) \geq 0$.

3.5 CPL

The structural rule of **CPL** is *Expansion*:

$$X \Leftarrow X;Y.$$

In natural deduction format:

X	$(i)\ A$	
$X;Y$	$(j)\ A$	i, Expansion

Things that can be proven in **CPL** but not in **RM**:

<u>Positive Paradox</u>	$\vdash P \rightarrow (Q \rightarrow P)$
<u>Negative Paradox</u>	$\vdash -P \rightarrow (P \rightarrow Q)$
	$\vdash P \rightarrow (Q \rightarrow Q)$
	$\vdash (P \wedge -P) \rightarrow Q$
	$\vdash P \rightarrow (Q \vee -Q)$

To see that these things cannot be proven in **RM**, we can use the Sugihara semantics. For example, for the third thing, $Q \rightarrow Q$ always gets assigned value $|Q|$. But then we can assign P any value that we wish, for example $f(P) = |Q| + 1$. Similarly, for Negative Paradox, for any fixed value of $-P$, we can adjust the value of Q to be as low or as high as we want. By doing so, we can adjust the value of $P \rightarrow Q$ as low as we want. And since the value of $-P$ is already fixed, we can make $-P \rightarrow (P \rightarrow Q)$ take a negative value. Hence $-P \rightarrow (P \rightarrow Q)$ is not a tautology of **RM**.

References

[AB75] Alan Ross Anderson and Nuel D. Belnap, Jr. *Entailment*. Princeton University Press, Princeton, NJ, 1975.

[Lem78] E. J. Lemmon. *Beginning logic*. Hackett Publishing Co., Indianapolis, Ind., revised edition, 1978. Edited by George W. D. Berry.

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[Sla90] John Slaney. A general logic. *Australasian Journal of Philosophy*, 68:74–89, 1990.