PHI 340: Midterm Exam Practice Problems

Note: We use the symbols $\wedge, \vee, \rightarrow, -$ for the classical propositional connectives.

1. Show that the modal propositional logic K is an extension of classical propositional logic. Use the definition of "extension" given in lecture. Discuss: Is K a proper extension of CPL?

2. Let $\models$ be the semantic consequence relation of CPL. We make the following definitions:

   Definition 1: If $\mathcal{X}$ is a set of sentences then $Cn(\mathcal{X}) = \{A : \mathcal{X} \models A\}$.

   Definition 2: A set $\mathcal{X}$ of sentences is a *deductive system* iff $\mathcal{X} = Cn(\mathcal{X})$.

   Definition 3: If $\mathcal{X}$ and $\mathcal{Y}$ are sets of sentences then

   $$\mathcal{X} \Cup \mathcal{Y} = Cn(\mathcal{X} \cup \mathcal{Y}), \qquad \mathcal{X} \Cap \mathcal{Y} = Cn(\mathcal{X} \cap \mathcal{Y}).$$

   Definition 4: If A is a sentence then $[A] = Cn(\{A\})$.

   (a) Prove that $\mathcal{X} \Cup \mathcal{Y}$ is a deductive system.

   (b) Prove that if $\mathcal{X}$ and $\mathcal{Y}$ are deductive systems then so is $\mathcal{X} \cap \mathcal{Y}$.

   (c) Prove that if $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are deductive systems such that $\mathcal{X} \subseteq \mathcal{Z}$ and $\mathcal{Y} \subseteq \mathcal{Z}$, then $\mathcal{X} \Cup \mathcal{Y} \subseteq \mathcal{Z}$.

   (d) Let A be an arbitrary sentence. What is contained in $[A] \Cup [-A]$?

   (e) What is contained in $[A] \Cap [-A]$?

   (f) Given: Set $\mathcal{X}$ contains either A or $-A$ for each sentence A. Question: Does it follow that $\mathcal{X}$ is a deductive system?

   (g) Show: $[A] \subseteq [B]$ iff $B \models A$.

   (h) Find a set theoretic equation with $[A \vee B]$ on one side, and $[A]$, $[B]$, plus one of $\Cap$ or $\Cup$ on the other side. Prove that your equation holds for all sentences A, B.

3. *Concerning the normal modal logics.* Recall that a T model is a triple $\langle \mathcal{W}, \mathcal{R}, v \rangle$ where $\mathcal{R}$ is a reflexive relation on $\mathcal{W}$.

   (a) Construct a T model in which $P \rightarrow \Box\Diamond P$ is true in every world, but $\Diamond P \rightarrow \Box\Diamond P$ is not true in every world.

(b) Assume for now that the syntax for K has only two atomic sentences, P and Q. We define a K model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, v \rangle$ as follows:

$$\mathcal{W} = \{w_1, w_2, w_3\}, \qquad \mathcal{R} = \{\langle w_1, w_1 \rangle, \langle w_2, w_3 \rangle\}.$$

i. Define a valuation v based on this model such that:

   for any two of these worlds, there is at least one sentence which does not have the same truth value in both of them.

ii. Define: $v_{\mathcal{M}}(A) = \{w \in \mathcal{W} : v(w, A) = 1\}$. We say that $v_{\mathcal{M}}(A)$ is the proposition expressed by A (in the model $\mathcal{M}$). Using the valuation you defined in the previous problem, identify the following propositions in the model $\mathcal{M}$:

$$v_{\mathcal{M}}(P) \quad v_{\mathcal{M}}(Q) \quad v_{\mathcal{M}}(-P) \quad v_{\mathcal{M}}(P \vee Q)$$
$$v_{\mathcal{M}}(\Box P) \quad v_{\mathcal{M}}(\Diamond P) \quad v_{\mathcal{M}}(\Diamond\Box P) \quad v_{\mathcal{M}}(\Box\Diamond P)$$

iii. Define what can be meant by:

   "Sentence A implies sentence B in model $\mathcal{M}$."

   Note well: This is a new notion that you are asked to define here — it is not the same as $\vdash$ or $\models$.

iv. Relying on this definition, identify which of the following sentences imply which in the model $\mathcal{M}$:

$$\Box P, \quad \Diamond P, \quad \Diamond\Box P, \quad \Box\Diamond P.$$

4. We define the language KW as follows: KW has the same syntax as K, and a model of KW is a triple $\langle \mathcal{W}, \mathcal{R}, v \rangle$ subject to the following two conditions:

   (a) $\mathcal{W}$ is finite.

   (b) $\mathcal{R}$ is transitive and irreflexive (i.e. there is no w such that $\langle w, w \rangle \in \mathcal{R}$).

   Show that in this language:

   (a) $\Box(\Box A \to A) \models \Box A$.

   (b) $\Box A \models \Box\Box A$.

   Also: construct a K tableau attempting to find a counterexample to:

$$\Box(\Box A \to A) \to \Box A, \Box A \models \Box\Box A.$$

   Comment on the outcome of your tableau.

5. Show that S5 and UA are equivalent languages — i.e. each is an extension of the other. Use the definition of "extension" given in lecture, with F the identity function. [Hint: Given an S5 valuation $v(w_0, \cdot)$, let $I(w_0) = \{w \in \mathcal{W} : Rw_0w\}$, use $I(w_0)$ to build a UA model and valuation.]