PHI 340 Handout

October 25, 2005

Conjecture 1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be languages with respective semantic consequence relations $\models_1$ and $\models_2$. If $\models_2$ is an extension of $\models_1$, then $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$.*

Consider first using the definition from *Possibilities and Paradox*: $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$ just in case all models of $\mathcal{L}_2$ are models of $\mathcal{L}_1$. Then the conjecture is *clearly* false: $\models_{\text{S5}}$ is an extension of $\models_{\text{UA}}$, and vice versa. But the models of S5 cannot be a subset of the models of UA, because they are different sorts of things: S5 models are triples $\langle \mathcal{W}, \mathcal{R}, v \rangle$, whereas UA models are doubles $\langle \mathcal{W}, v \rangle$. So what could we mean by saying that S5 models are a subset of UA models? One could consider a UA model to be a triple, where the accessibility relation $\mathcal{R}$ is just the universal relation $\mathcal{W} \times \mathcal{W}$. But in that case, not all S5 models are UA models, because not all S5 models have a universal accessibility relation. (However, your intuition is still correct: there is a natural correspondence between S5 valuations and UA valuations, and that is sufficient to show that $\models_{\text{S5}}$ coincides with $\models_{\text{UA}}$. *Homework*: Prove the previous sentence.)

Consider now using the definition from the handout. Then the conjecture is **False**. We show that it is false by providing a counterexample.

Counterexample. We define the language $\mathcal{L}_1$ to have one atomic sentence P, and no connectives. The language $\mathcal{L}_2$ has exactly the same syntax.

We define the set of valuations of $\mathcal{L}_1$ to be a singleton set $\{\alpha\}$. We stipulate that α satisfies *no sentences*. In other words, the relation "x satisfies A" is just the empty relation: $\emptyset \times \emptyset$.

We define the set of valuations of $\mathcal{L}_2$ to be a singleton set $\{\omega\}$. We stipulate that ω satisfies *all sentences*. In other words, the relation "x satisfies A" is $\{\omega\} \times \mathcal{S}$, where $\mathcal{S}$ is the set of all sentences.

We now establish that $\models_2$ is an extension of $\models_1$. This, in fact, is obvious: Since $\mathcal{L}_2$ has only one valuation, and it satisfies all sentences, it follows that $\mathcal{X} \models_2 A$ for *any* set $\mathcal{X}$ of sentences, and for *any* sentence A. In other words, $\models_2$ is just the relation $\mathcal{P}(\mathcal{S}) \times \mathcal{S}$. So, any consequence relation (on this syntax) is a subset of $\models_2$.

We now establish that $\mathcal{L}_2$ is not an extension of $\mathcal{L}_1$. Suppose for reductio that $\mathcal{L}_2$ is an extension of $\mathcal{L}_1$. Then there are functions $F : \mathcal{S}_1 \to \mathcal{S}_2$ and $F^* : \mathcal{V}_2 \to \mathcal{V}_1$ such that

$$F^*(x) \text{ satisfies } A \quad \Longleftrightarrow \quad x \text{ satisfies } F(A).$$

Now, there is only one sentence P of each language, so we must have $F(P) = P$. Furthermore, there is only one valuation of each language, so we must have $F^*(\omega) = \alpha$. Thus, we must have

$$\alpha \text{ satisfies } P \quad \Longleftrightarrow \quad \omega \text{ satisfies } P.$$

But ω satisfies P, whereas α does not satisfy P, a contradiction. Therefore, $\mathcal{L}_2$ is not an extension of $\mathcal{L}_1$. $\square$

Suppose that we change the example so that the languages each have one unary connective $\neg$. Does the proof still go through?

Yes: Let $F : \mathcal{S}_1 \to \mathcal{S}_2$. Then $F(P) = \neg \cdots \neg P$, with some number (possibly zero) of negation symbols on the front of P. But now our biconditional says:

$$\alpha \text{ satisfies } P \quad \Longleftrightarrow \quad \omega \text{ satisfies } \neg \cdots \neg P.$$

But ω satisfies $\neg \cdots \neg P$, whereas α does not satisfy P, a contradiction.

What if we give each language the full syntax of CPL? The result still goes through.

For Further Thought: The languages defined above are "pathological" in some obvious ways. For example, it seems odd to have a language with no valuations that satisfy any sentences. But our definition of a language allows such pathologies.

So maybe we should rule out languages such as those above. What do you think would be natural further requirements to impose on a language?

What could we require in the way of there being “enough” valuations? Are there further conditions that would guarantee that the above Conjecture is validated?