# PHI 340 Definitions

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**Definition 1.** Let $A$ and $B$ be sets. A *function* $f$ from $A$ to $B$ is a relation on $A \times B$ such that: for each $x \in A$, there is a *unique* $y \in B$ such that $\langle x, y \rangle \in f$. In this case, we can write $f(x)$ to denote that unique element of $B$ to which $x$ bears the relation. We sometimes use $f : A \to B$ to denote a function from $A$ to $B$.

**Definition 2.** $f$ is said to be *one-to-one* or *injective* just in case:

$$\forall x \forall y[(f(x) = f(y)) \implies x = y]$$

**Definition 3.** $f$ is said to be *onto* or *surjective* just in case:

$$(\forall y \in B)(\exists x \in A)[f(x) = y]$$

**Definition 4.** A *sentential syntax* consists of:

1. Vocabulary

   (a) A set **At** of atomic sentences (we usually denote these by lowercase letters, starting with $p$);

   (b) For each $n = 1, 2, \ldots,$, a set $C_n$ of $n$-ary connectives;

   (c) Punctuation marks ")" and "(";

   (d) The preceding three sets are disjoint.

2. The set **Snt** of sentences is defined inductively:

   (a) All elements of **At** are in **Snt**;

   (b) If $\varphi$ is an $n$-ary connective and $a_1, \ldots, a_n \in$ **Snt**, then $\varphi(a_1, \ldots, a_n) \in$ **Snt**;

   (c) Nothing is in **Snt** except things generated by the preceding two clauses.

> We usually denote sentences by lowercase letters $a, b, c, d$. So, note: $a$ could denote the sentence $p$ or the sentence $p \wedge q$, etc.. But $p$ *is* a sentence — it does not denote some other sentence.

So, a syntax is said to be a "triple" $\langle \mathbf{At}, C, \mathbf{Snt} \rangle$, where $C = \cup_n C_n$.

**Definition 5.** A *valuation* of a syntax $\langle \mathbf{At}, C, \mathbf{Snt} \rangle$ is a function from **Snt** into some value set $V$.

**Definition 6.** A *language* $\mathcal{L}$ consists of:

1. A syntax $\langle \mathbf{At}, C, \mathbf{Snt} \rangle$;
2. A semantics, which includes:
    (a) A set $\mathrm{Val}(\mathcal{L})$ of valuations (called the *admissible valuations* of $\mathcal{L}$);
    (b) A relation "**true**" between admissible valuations and sentences (called *satisfaction* or *making true*).

**Definition 7.** A sentence $a$ of $\mathcal{L}$ is *valid*, written $\models a$, just in case:

$$\forall \omega \in \mathrm{Val}(\mathcal{L})[\omega \text{ satisfies } a]$$

**Definition 8.** A sentence $A$ of $\mathcal{L}$ is *satisfiable* just in case:

$$\exists \omega \in \mathrm{Val}(\mathcal{L})[\omega \text{ satisfies } a]$$

**Definition 9.** *$a$ implies $b$* in $\mathcal{L}$, written $a \models b$, just in case:

$$\forall \omega \in \mathrm{Val}(\mathcal{L})[\omega \text{ satisfies } a \implies \omega \text{ satisfies } b]$$

**Definition 10.** A subset $X$ of sentences is *satisfiable* in $\mathcal{L}$ just in case:

$$(\exists \omega \in \mathrm{Val}(\mathcal{L}))(\forall a \in X)[\omega \text{ satisfies } a].$$

**Definition 11.** A subset $X$ of sentences *implies* a sentence $a$, written $X \models a$, just in case:

$$(\forall \omega \in \mathrm{Val}(\mathcal{L}))[\omega \text{ satisfies } X \implies \omega \text{ satisfies } a]$$

**Definition 12.** A *logical system* for a language $\mathcal{L}$ is a relation $\vdash$ of "derivability from" (note: in reverse order!) between elements of **Snt** and <u>finite</u> subsets of **Snt**. So, $a_1, \ldots, a_n \vdash b_1$ means: "there is a proof with $a_1, \ldots, a_n$ as premises, and $b$ as a conclusion." This relation should, at the very least, be "recursively enumerable."

**Definition 13.** A sentence $a$ is a *theorem* of the logical system $K$ iff $\vdash a$.

**Definition 14.** $\vdash$ is *statement sound* relative to $\models$:

$$\vdash a \quad \Longrightarrow \quad \models a$$

**Definition 15.** $\vdash$ is *statement complete* relative to $\models$:

$$\vdash a \quad \Longleftarrow \quad \models a$$

**Definition 16.** $\vdash$ is *argument sound* relative to $\models$:

$$X \vdash a \quad \Longrightarrow \quad X \models a, \qquad \text{for finite } X.$$

**Definition 17.** $\vdash$ is *argument complete* relative to $\models$:

$$X \vdash a \quad \Longleftarrow \quad X \models a, \qquad \text{for finite } X.$$

**Definition 18.** $\vdash$ is *strongly complete* relative to $\models$:

$$X \models a \quad \Longrightarrow \quad [(\exists b_1, \ldots, b_n \in X) \text{ s.t. } b_1, \ldots, b_n \vdash a]$$

**Definition 19.** Let $\omega : \mathbf{Snt} \to V$ be a valuation. A *base* for $\omega$ consists of operators $\mathbb{C}$ on $V$ and a function $\odot_{\_} : C \to \mathbb{C}$ such that

$$\omega(\varphi(a_1, \ldots, a_n)) = \odot_\varphi(\omega(a_1), \ldots, \omega(a_n)),$$

for all connectives $\varphi \in C$.

**Definition 20.** A language $\mathcal{L}$ is said to be *compact* if for any set $X$ of sentences: if all finite subsets of $X$ are satisfiable, then $X$ is satisfiable.