ECO 199 – GAMES OF STRATEGY
Spring Term 2004 – February 26
MIXED STRATEGIES – ZERO-SUM GAMES

MIXED STRATEGY – random choice, with specified probabilities,
from the originally specified or "pure" strategies
These work differently in zero-sum and non-zero-sum games
In zero-sum games, reason is to keep the other guessing,
when any systematic action would be exploited by the other
to his benefit, and therefore to your cost

EXPECTED PAYOFF of mixed strategy – *Sum* over all pure strategies
of the *products* of Probability and Payoff

BASIC 2-by-2 GAME

Robbers choose hiding place; Cops choose search focus
Robbers' payoffs are escape probabilities in percent
Cops' payoffs (not shown) negative of this (or 100 minus this)
Or think of Cops as trying to minimize Robbers' payoffs
This is special feature of zero-sum games

		Cops		
		City	Suburb	Min
Robbers	City	20	70	20
	Suburb	80	30	30
	Max	80	70	

Check directly that there is no Nash equilibrium in pure strategies

Also another test for zero-sum games
Robbers' Maxi-Min = 30 , Cops' Mini-Max= 70
Maxi-Min < Mini-Max , no pure strategy Nash equilibrium

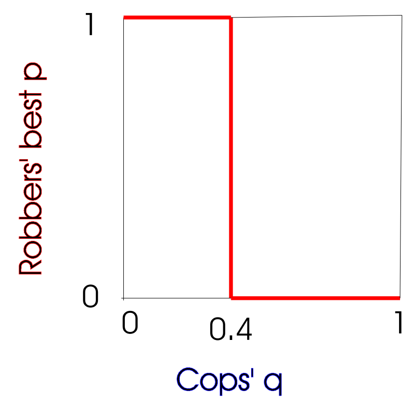
BEST RESPONSE ANALYSIS

Mixed strategy is one kind of continuous strategy:
Probability is the continuous variable, ranging from 0 to 1.
In the Cops-Robbers example
For Robbers – "p-mix", choosing C with probability p, S with (1-p)
For Cops – "q-mix", choosing C with probability q, S with (1-q)

ROBBERS' BEST RESPONSE

		Cops		
		City	Suburb	C:q, S:1-q
Robbers	City	20	70	20 q + 70 (1-q)
	Suburb	80	30	80 q + 30 (1-q)

Robbers' best p as function of Cops' q



Pure C (p=1) better than pure S (p=0) if
$20 q + 70 (1-q) > 80 q + 30 (1-q)$
$60 q < 40 (1-q)$, $100 q < 40$, $q < 0.4$

Robbers' expected payoff for general p
$= p \; [20 q + 70 (1-q)]$
$+ (1-p) \; [80 q + 30 (1-q)]$
varies linearly with p
Therefore in same case ($q < 0.4$),
$p = 1$ is also better than any other p in the range from 0 to 1
That is, $p = 1$ (pure C) is the Robbers' best response if $q < 0.4$

Conversely, pure S ($p = 0$) is Robbers' best response if $q > 0.4$

All values of p between 0 and 1 are equally good if $q = 0.4$

Robbers' best response "curve" is a step-function

COPS' BEST RESPONSE

		Cops	
		City	Suburb
Robbers	City	20	70
	Suburb	80	30
	C:p, S:1-p	20 p + 80 (1-p)	70 p + 30 (1-p)

Cops' best q as function of Robbers' p

Pure C (q=1) better than pure S (q=0) if

20 p + 80 (1-p) < 70 p + 30 (1-p)

(remember these are Robbers' payoffs and Cops want small numbers)

50 (1-p) < 50 p, 100 p > 50, p > 0.5

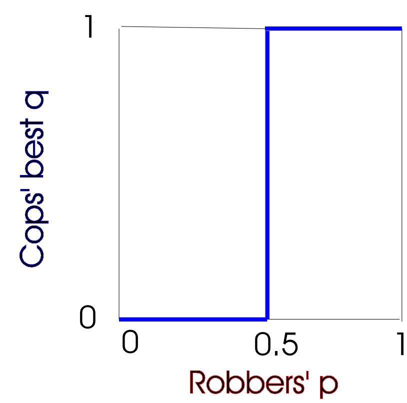
Using same reasoning as above,

Cops' best response is

pure C (q=1) if p > 0.5, pure S (q=0) if p < 0.5

Everything equally good if p = 0.5

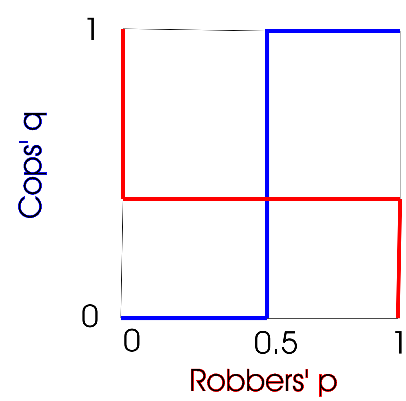
Best response "curve" is the step-function



NASH EQUILIBRIUM IN p AND q

Put the two best response "curves" together (switching axes of one)

Intersection of best response curves is mixed strategy Nash equilibrium



Interpretation of correct beliefs -

If each believes that the other is mixing in the specified proportions

Then each is indifferent between his own C and S, and mixing in the specified proportions is as good as anything

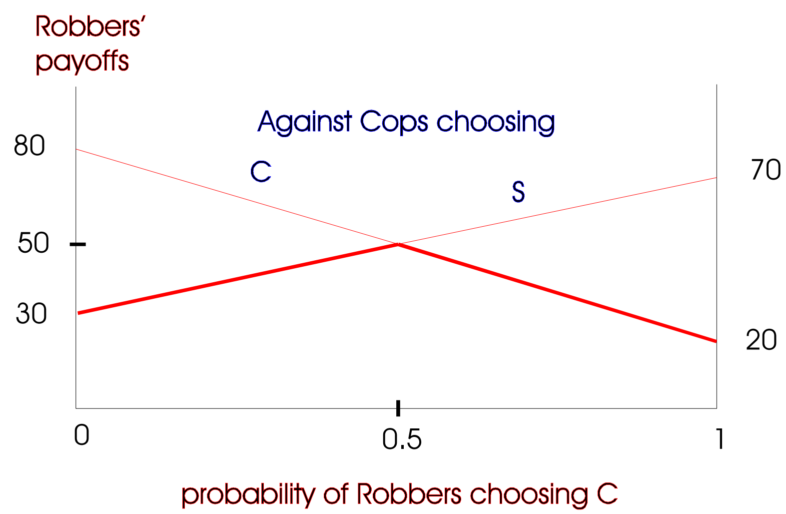
So these choices *can* sustain correct beliefs

No other mixtures can be similarly self-sustaining

MIXING IMPROVES ROBBERS' MAXI-MIN

		Cops		
		City	Suburb	Min
Robbers	City	20	70	20
	Suburb	80	30	30
	C:p, S:1-p	20 p + 80 (1-p)	70 p + 30 (1-p)	See below

Robbers' payoffs as functions of their own p in the mixture "p-mix"
Two different lines corresponding to Cops' choice of C or S
For each p, the worst for Robbers is the min of these two lines
or the "lower envelope" of the lines, shown thicker
Robbers choose the p that gives the max of these mins



Robbers' Maxi-Min payoff = 50, achieved when p = 0.5
This is > the Maxi-Min with pure strategies, namely 30
So mixing improves the Robbers' Maxi-Min
And Nash equilibrium mixture gets them best Maxi-Min
That is, best protects them against the worst the Cops can do

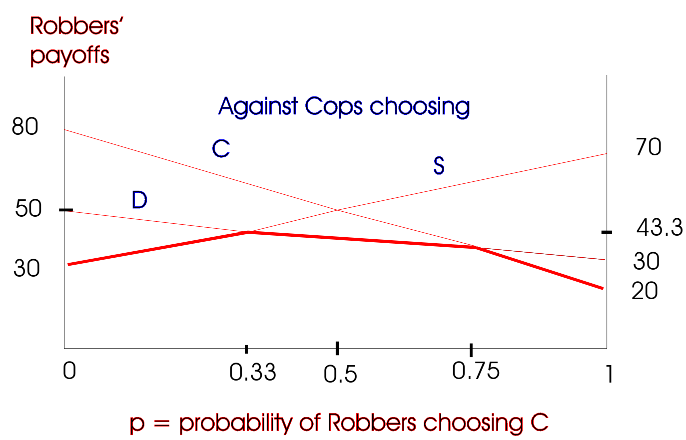
Similarly mixing improves Cops' Mini-Max, 50 attained when q = 0.4
This is < their mini-max of 70 attainable with pure strategies
And Robbers' Maxi-Min = Cops' Mini-Max !

von Neumann and Morgenstern's "minimax theorem" :
In zero-sum games with mixed strategies, Maxi-Min = Mini-Max
And this gives a (Nash) equilibrium

WHEN COPS HAVE THIRD STRATEGY

		Cops		
		City	Suburb	Divide force
Robbers	City	20	70	30
	Suburb	80	30	50
	C:p, S:1-p	20p+80(1-p)	70p+30(1-p)	30p+50(1-p)

Robber's choice of p maxes the min of these three lines



Result – p = 0.33, expected payoff 43.3 against Cops' D or S
This mix would yield 60 against Cops' C
So Cops don't use C in their mix. Can find their q-mix of D and S
Result – S:0.33, D:0.67, expected payoff 43.3 against Robbers' C or S

In general case, number of active strategies for either player no more than the smaller of the numbers of pure strategies for the two
Two exceptional cases
(1) if D, C and S lines all pass through one point, then all three strategies can be active in equilibrium; mix varies over range
(2) if D line is flat, Cops' equilibrium p can vary over a range

Soccer penalty kick example - solution found using Gambit:

Case 1 - All strategies active		Goalie: Left	Goalie: Center	Goalie: Right	Kicker's prob	Result against Goalie's mix
Kicker	Left	45	90	90	0.355	75.4
Kicker	Center	85	0	85	0.188	75.4
Kicker	Right	95	95	60	0.457	75.4
Goalie's Prob		0.325	0.113	0.561		
Result against Kicker's mix		75.4	75.4	75.4		

Case 2 - Some strategies inactive		Goalie: Left	Goalie: Center	Goalie: Right	Kicker's prob	Result against Goalie's mix
Kicker	Left	45	90	90	0.4375	73.13
Kicker	Center	70	0	70	0	70.00
Kicker	Right	95	95	60	0.5625	73.13
Goalie's Prob		0.375	0	0.625		
Result against Kicker's mix		73.13	92.8	73.13		

Principle of "complementary slackness" – Against other's equilibrium mix,
All strategies active in your mix fare equally well
All inactive strategies fare worse than this (in exceptional cases, =)