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A Rejection probabilities using the t-ratio

A.1 Notation for Appendix

This appendix collects proofs of the results claimed in the text. In the interest of being self-contained, we recapitulate our general notation. For a representative observation, the model is

$$\begin{aligned} Y &= X\beta + u \\ X &= Z\pi + v \end{aligned}$$

for an outcome Y, a single endogenous regressor X, and a single instrument Z. While suppressed, the model above allows for constants and covariates, as long as we interpret the triple (X,Y,Z) as the residuals from a regression on any covariates W and a constant.[32]

The IV estimator itself is $\widehat{\beta} = \mathbf{Z}'\mathbf{Y}/\mathbf{Z}'\mathbf{X}$, where **bold** denotes a vector, the first-stage estimator is $\widehat{\pi} = \mathbf{Z}'\mathbf{X}/\mathbf{Z}'\mathbf{Z}$, and the reduced-form estimator is $\widehat{\pi\beta} = \mathbf{Z}'\mathbf{Y}/\mathbf{Z}'\mathbf{Z}$. Note that we write the reduced-form coefficient as $\widehat{\pi\beta}$ because the reduced-form coefficient is numerically equal to the product of $\widehat{\pi}$ and $\widehat{\beta}$. The IV fitted residual is $\widehat{u} = Y - X\widehat{\beta}$, and we analogously write $\widehat{v}$ and $\widehat{\varepsilon}$ for the fitted residual from the first-stage and reduced-form regressions; population analogues we denote by v and ε, respectively.

Throughout we will be examining HAC variance estimators. Consider, for example, the first-stage estimated variance, given by

$$\widehat{V}_N(\widehat{\pi}) \quad = \quad (\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{v})(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{\widehat{V}(Z\widehat{v})}{(\mathbf{Z}'\mathbf{Z})^2}$$

In the display above, we are using the notation $\widehat{V}_N(\cdot)$ to convey the estimated variance for a parameter. In contrast, we write $\widehat{V}(Z\widehat{v})$ (without a subscript of N) as a unifying notation for the "meat" of the sandwich variance estimator in order to cover the multitude of approaches to variance estimation encountered in applications: homoskedastic standard errors, heteroskedasticity-robust standard errors, clustered standard errors, two-way clustered standard errors, time-series approaches such as

[32] Algebra and the partitioned inverse theorem shows that ignoring covariates and constants leaves point estimates and fitted residuals (and thus variance estimators) the same, as long as we reinterpret the trio (X,Y,Z) as the residuals from a regression of each of them on W. This is a simple extension of the same point made in the regression context by Theorem 4.1 of Lovell (1963) and is an application of Theorem 6.1 of Newey and McFadden (1994).

Newey-West (1987), or yet other HAC approaches.[33] Moreover, and slightly less standardly, we will use a similar notation for covariance.

Beginning with Lemma 5, below, we will invoke three high-level assumptions that we now state.

Assumption 1 (Asymptotically Finite First-Stage)**.** $\pi_N\sqrt{\mathbf{Z}'\mathbf{Z}} \xrightarrow{p} \pi_{ZZ}$.

Note that in the main text, we wrote the true first stage parameter as π, but here we clarify that in a weak IV framework, the asymptotic sequence is one in which the parameter π shrinks towards zero. In this appendix, we clarify this with notation by writing π_N, where the parameter sequence satisfies Assumption 1.

Assumption 2 (Asymptotic Normality)**.** $\frac{1}{\sqrt{\mathbf{Z}'\mathbf{Z}}}\begin{pmatrix}\mathbf{Z}'\varepsilon \\ \mathbf{Z}'\mathbf{v}\end{pmatrix} \xrightarrow{d} N(\mathbf{0},\Sigma)$, with $\Sigma \equiv \begin{pmatrix}\sigma_\varepsilon^2 & \sigma_{\varepsilon v} \\ \sigma_{\varepsilon v} & \sigma_v^2\end{pmatrix}$.

Assumption 2 is sufficient to imply that the first-stage and reduced-form estimators are consistent and asymptotically normal.

Assumption 3 (Consistent Variance and Covariance Estimators)**.**

$$\begin{aligned}
\widehat{V}(Z\widehat{\varepsilon})/N - V\left(\mathbf{Z}'\varepsilon/\sqrt{N}\right) &\xrightarrow{p} 0 \\
\widehat{V}(Z\widehat{v})/N - V\left(\mathbf{Z}'\mathbf{v}/\sqrt{N}\right) &\xrightarrow{p} 0 \\
\widehat{C}(Z\widehat{\varepsilon}, Z\widehat{v})/N - C\left(\mathbf{Z}'\varepsilon/\sqrt{N}, \mathbf{Z}'\mathbf{v}/\sqrt{N}\right) &\xrightarrow{p} 0
\end{aligned}$$

[33]For example, if the variance matrix of the errors is taken to be spherical, we would use

$$\widehat{V}(Z\widehat{v}) = \widehat{\sigma}^2\left(\sum_i Z_i^2\right)$$

where $\widehat{\sigma}^2 = \frac{1}{N}\sum_i \widehat{v}_i^2$, and the sum is over the data. In contrast, if the errors were taken to be heteroskedastic, then we would use

$$\widehat{V}(Z\widehat{v}) = \sum_i Z_i^2 \widehat{v}_i^2$$

If a clustered approach is taken, with groups indexed by j and observations within group indexed by i, we would instead use

$$\widehat{V}(Z\widehat{v}) = \sum_j \mathbf{Z}_j\widehat{\mathbf{v}}_j\widehat{\mathbf{v}}_j'\mathbf{Z}_j'$$

where $\mathbf{Z}_j$ is the stack of instruments for group j, $\widehat{\mathbf{v}}_j$ is the stack of estimated residuals for group j, and the sum is over the clusters j. For two-way clustering (e.g., Cameron, Gelbach, and Miller 2011) or time-series approaches (e.g., Newey-West 1987), the results are *mutatis mutandis.*

Assumption 3 simply states that the variance estimators being employed are consistent. In the main text, we write $V^*_{Z\varepsilon}$ in place of $\lim_{N\to\infty} V(\mathbf{Z}'\varepsilon/\sqrt{N})$ and analogously for v. This is to simplify exposition; we do not necessarily require that limit to exist, but instead require the milder form stated in Assumption 3.

A.2 Relationship between IV and reduced-form variance estimators

Lemma 1 (Relationship Between IV and Reduced-Form Variance Estimators)**.**

$$\widehat{V}_N(\hat{\beta}) \;=\; \frac{1}{\widehat{\pi}^2}\left\{\widehat{V}_N(\widehat{\pi\beta}) - 2\widehat{\beta}\widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}\,) + \widehat{\beta}^2\widehat{V}_N(\widehat{\pi})\right\}$$

PROOF: In the just-identified case with a single endogenous regressor, the standard formula for the estimated IV variance reduces so that

$$\widehat{V}_N(\hat{\beta}) \;=\; (\mathbf{Z}'\mathbf{X})^{-1}\widehat{V}(Z\widehat{u})(\mathbf{X}'\mathbf{Z})^{-1} = \widehat{V}(Z\widehat{u})/(\mathbf{Z}'\mathbf{X})^2$$

Similarly, the estimated variances and covariances for the reduced-form coefficient $\widehat{\pi\beta}$ and the first-stage coefficient $\widehat{\pi}$ are given by

$$\begin{aligned}\widehat{V}_N(\widehat{\pi\beta}) &= \widehat{V}(Z\widehat{\varepsilon})/(\mathbf{Z}'\mathbf{Z})^2\\ \widehat{V}_N(\widehat{\pi}) &= \widehat{V}(Z\widehat{v})/(\mathbf{Z}'\mathbf{Z})^2\\ \widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}) &= \widehat{C}(Z\widehat{\varepsilon},Z\widehat{v})/(\mathbf{Z}'\mathbf{Z})^2\end{aligned}$$

where $\widehat{\varepsilon} \equiv Y - Z\widehat{\pi\beta}$ and $\widehat{v} \equiv X - Z\widehat{\pi}$ are the reduced-form and first-stage fitted residuals, respectively. For a representative observation we have $\widehat{\varepsilon} = Y - X\widehat{\beta} + X\widehat{\beta} - Z\widehat{\pi\beta} = \hat{u} + \hat{v}\widehat{\beta}$, and since $\widehat{\beta}$ does not vary by observation, we have $\widehat{u}\widehat{u}' = \widehat{\varepsilon}\widehat{\varepsilon}' - 2\widehat{\beta}\widehat{\varepsilon}\widehat{v}' + \widehat{\beta}^2\widehat{v}\widehat{v}'$ which in turn implies that the middle factors of the various sandwich variance estimates are all functionally related:

$$\widehat{V}(Z\widehat{u}) \;=\; \widehat{V}(Z\widehat{\varepsilon}) - 2\widehat{\beta}\widehat{C}(Z\widehat{\varepsilon},Z\widehat{v}) + \widehat{\beta}^2\widehat{V}(Z\widehat{v})$$

Putting these results together, we see that

$$\begin{aligned}\widehat{\pi}^2\widehat{V}_N(\widehat{\beta}) &= \left(\frac{\mathbf{Z}'\mathbf{X}}{\mathbf{Z}'\mathbf{Z}}\right)^2\widehat{V}_N(\widehat{\beta}) = \frac{\widehat{V}(Z\widehat{u})}{(\mathbf{Z}'\mathbf{Z})^2} = \frac{\widehat{V}(Z\widehat{\varepsilon})}{(\mathbf{Z}'\mathbf{Z})^2} - 2\widehat{\beta}\frac{\widehat{C}(Z\widehat{\varepsilon},Z\widehat{v})}{(\mathbf{Z}'\mathbf{Z})^2} + \widehat{\beta}^2\frac{\widehat{V}(Z\widehat{v})}{(\mathbf{Z}'\mathbf{Z})^2}\\ &= \widehat{V}_N(\widehat{\pi\beta}) - 2\widehat{\beta}\widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}) + \widehat{\beta}^2\widehat{V}_N(\widehat{\pi})\end{aligned}$$

and the result follows after dividing both sides of the above by $\widehat{\pi}^2$. ☒

Lemma 2 (t-test for IV)**.**

$$\widehat{t} \equiv \widehat{t}(\beta_0) = \frac{\widehat{\beta} - \beta_0}{\hat{se}(\hat{\beta})} = \frac{|\widehat{\pi}|(\widehat{\beta} - \beta_0)}{\sqrt{\hat{V}_N(\widehat{\pi\beta}) - 2\hat{\beta}\hat{C}_N(\widehat{\pi\beta}, \hat{\pi}) + \widehat{\beta}^2\hat{V}_N(\hat{\pi})}}$$

where $\hat{se}(\cdot) = \sqrt{\widehat{V}_N(\cdot)}$ *is notation for the estimated standard error of a parameter.*

PROOF: The result follows immediately from Lemma 1. ☒

Remark (Dependence on β_0)**.** Note that while we follow standard econometric practice and write $\widehat{t}$ for the estimated t-statistic, it of course is true that the t-statistic depends on the parameter value being tested (i.e., β_0). For statistics other than the t-statistic, we will emphasize dependence on β_0 by writing them as functions of β_0. Note that in our notation, β_0 is not necessarily the true parameter value, but could also be a hypothesized (but false) parameter value, i.e., there is no reason to assume $\beta = \beta_0$.

Remark (Form of the F statistic)**.** In the just-identified context, we have

$$\widehat{F} \equiv \frac{\widehat{\pi}^2}{\widehat{V}(\widehat{\pi})} = \frac{\left((\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}\right)^2}{(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{v})(\mathbf{Z}'\mathbf{Z})^{-1}} = \frac{(\mathbf{Z}'\mathbf{X})^2}{\widehat{V}(Z\widehat{v})}$$

and

$$\widehat{f} \quad = \quad \frac{\widehat{\pi}}{\sqrt{\widehat{V}(\widehat{\pi})}} = \frac{\mathbf{Z}'\mathbf{X}}{\sqrt{\widehat{V}(Z\widehat{v})}}$$

where $\widehat{f}$ is the signed t-test on the exclusion of the instrument in the first-stage regression. Note that in this context $\widehat{F}$ is the same as the "effective F statistic" described in Olea and Pflueger (2013).

A.3 t-ratio form of Anderson-Rubin statistic

Lemma 3 (Similarity of the AR-statistic and the t-statistic)**.** *The AR test statistic can be written as in a form that is similar to the formula for the t-statistic for the structural parameter, but with a variance estimator that imposes the null:*

$$\widehat{t}_{AR}(\beta_0) \quad = \quad \frac{\widehat{\pi}(\widehat{\beta} - \beta_0)}{\hat{\text{se}}(\widehat{\pi}(\widehat{\beta} - \beta_0))} = \frac{\widehat{\pi}(\widehat{\beta} - \beta_0)}{\sqrt{\widehat{V}_N(\widehat{\pi\beta}) - 2\beta_0\widehat{C}_N(\widehat{\pi\beta}, \widehat{\pi}) + \beta_0^2\widehat{V}_N(\widehat{\pi})}}$$

PROOF: For any given approach to variance estimation, the *AR* test of the null hypothesis $\beta = \beta_0$ can be obtained by: (1) forming the residual $u_0 = Y - X\beta_0$, (2) regressing u_0 on Z, and (3) using an F test to test the null hypothesis that the coefficient on Z in that regression is zero, where the F test adopts the desired approach to inference for the original IV model.[34]

This gives rise to concepts of the *AR* coefficient and the *AR* standard error, by which we mean simply the coefficient and standard error from the regression in that third step, respectively. Consider each in turn. The *AR* coefficient is

$$\frac{\mathbf{Z}'(\mathbf{Y}-\mathbf{X}\beta_0)}{\mathbf{Z}'\mathbf{Z}} \quad = \quad \widehat{\pi\beta} - \widehat{\pi}\beta_0 = \widehat{\pi}(\widehat{\beta}-\beta_0)$$

where the last result follows since the reduced-form coefficient $\widehat{\pi\beta}$ is the product of the first-stage coefficient $\widehat{\pi}$ and the estimated structural parameter $\widehat{\beta}$. The *AR* standard error can be thought of in two ways. First, and more standardly, let $\widehat{u}_0$ denote the fitted *AR* regression residual. Then the estimated *AR* standard error is the square root of

$$(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{u}_0)(\mathbf{Z}'\mathbf{Z})^{-1} = \frac{\widehat{V}(Z\widehat{u}_0)}{(\mathbf{Z}'\mathbf{Z})^2}$$

Second, since the *AR* coefficient is a linear combination of the reduced-form and first-stage coefficients, as shown above, it is the square root of

$$\widehat{V}_N(\widehat{\pi\beta}) - 2\beta_0\widehat{C}_N(\widehat{\pi\beta},\widehat{\pi}) + \beta_0^2\widehat{V}_N(\widehat{\pi})$$

The lemma follows from the second result. We will use the first characterization of the *AR* standard error in Lemma 4, below. ⊠

In light of Lemmas 2 and 3, it is not surprising that there is a numerical equivalence allowing one to obtain $\widehat{t}$ from $\widehat{t}_{AR}(\beta_0)$ and other quantities, the subject to which we next turn.

[34] If covariates are part of the model, then the degrees of freedom for the F test should be adjusted to reflect the dimension of the covariates W that were partialled out in the first step.

A.4 From $\hat{t}_{AR}(\beta_0)$ to $\hat{t}$

Lemma 4 (Dependence of $\hat{t}$ on $\hat{t}_{AR}(\beta_0)$, $\widehat{\rho}(\beta_0)$ and $\widehat{f}$).

$$\hat{t}^2 = \frac{\hat{t}^2_{AR}(\beta_0)}{1-2\widehat{\rho}(\beta_0)\frac{\hat{t}_{AR}(\beta_0)}{\widehat{f}}+\frac{\hat{t}^2_{AR}(\beta_0)}{\widehat{f}^2}}$$

where

$$\widehat{\rho}(\beta_0) \equiv \frac{\widehat{C}(Z\widehat{u}_0, Z\widehat{v})}{\sqrt{\widehat{V}(Z\widehat{u}_0)}\sqrt{\widehat{V}(Z\widehat{v})}}$$

and as emphasized in the second remark after Lemma 1, $\widehat{f}$ is the t-ratio test on the exclusion of the instrument in the first-stage regression, i.e., $\widehat{F}=\widehat{f}^2$. We emphasize that Lemma 4 is not an approximation, but instead a numerical equivalence.

PROOF: We first note that the structural residual combines the *AR* regression residual with the first-stage residual. To get there, note that the *AR* residual is

$$\widehat{u}_0 = Y - X\beta_0 - Z\widehat{\pi}(\widehat{\beta}-\beta_0)$$

That is, the outcome for the *AR* regression is $Y - X\beta_0$ and the predicted value is $Z\widehat{\pi}(\widehat{\beta}-\beta_0)$. Then we add and subtract $X\beta_0$ and $Z\widehat{\pi}(\widehat{\beta}-\beta_0)$ from $\widehat{u}$ to obtain

$$\begin{aligned} \widehat{u} &= Y - X\widehat{\beta} = Y - X\widehat{\beta} + X\beta_0 - X\beta_0 + Z\widehat{\pi}(\widehat{\beta}-\beta_0) - Z\widehat{\pi}(\widehat{\beta}-\beta_0) \\ &= \widehat{u}_0 - \widehat{v}(\widehat{\beta}-\beta_0) \end{aligned}$$

As in the proof of Lemma 1, and for the same reasons, we can use the result above to re-write the meat of the IV variance estimate:

$$\widehat{V}(Z\widehat{u}) = \widehat{V}(Z\widehat{u}_0) - 2(\widehat{\beta}-\beta_0)\widehat{C}(Z\widehat{u}_0, Z\widehat{v}) + (\widehat{\beta}-\beta_0)^2\widehat{V}(Z\widehat{u}_0)$$

Next, note that $\hat{t}^2$ and $\hat{t}^2_{AR}(\beta_0)$ differ only to the extent $\widehat{V}(Z\widehat{u})$ and $\widehat{V}(Z\widehat{u}_0)$ differ:

$$\begin{aligned} \hat{t}^2 &= \frac{(\widehat{\beta}-\beta_0)^2}{\widehat{V}(Z\widehat{u})/(\mathbf{Z}'\mathbf{X})^2} \\ \hat{t}^2_{AR}(\beta_0) &= \frac{\widehat{\pi}^2(\widehat{\beta}-\beta_0)^2}{\widehat{V}(Z\widehat{u}_0)/(\mathbf{Z}'\mathbf{Z})^2} = \frac{(\widehat{\beta}-\beta_0)^2}{\widehat{V}(Z\widehat{u}_0)/(\mathbf{Z}'\mathbf{X})^2} \end{aligned}$$

Then, using the above result on $\widehat{V}(Z\widehat{u})$, we obtain

$$\begin{aligned}\frac{\hat{t}^2}{\hat{t}_{AR}^2(\beta_0)} &= \frac{\widehat{V}(Z\widehat{u}_0)}{\widehat{V}(Z\widehat{u})} = \frac{\widehat{V}(Z\widehat{u}_0)}{\widehat{V}(Z\widehat{u}_0) - 2(\widehat{\beta}-\beta_0)\widehat{C}(Z\widehat{u}_0, Z\widehat{v}) + (\widehat{\beta}-\beta_0)^2\widehat{V}(Z\widehat{v})} \\ &= \frac{1}{1 - 2\widehat{\rho}(\beta_0)(\widehat{\beta}-\beta_0)\sqrt{\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u}_0)}} + (\widehat{\beta}-\beta_0)^2\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u}_0)}}\end{aligned}$$

Finally, note that

$$\frac{\widehat{t}_{AR}(\beta_0)}{\widehat{f}} = \frac{\widehat{\pi}(\widehat{\beta}-\beta_0)}{\sqrt{\widehat{V}(Z\widehat{u}_0)}\Big/(\mathbf{Z}'\mathbf{Z})}\frac{\sqrt{\widehat{V}(Z\widehat{v})}\Big/(\mathbf{Z}'\mathbf{Z})}{\widehat{\pi}} = (\widehat{\beta}-\beta_0)\sqrt{\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u}_0)}}$$

and the result follows. ⊠

A.5 From $\widehat{t}$ to $\widehat{t}_{AR}^2$

Lemma 5 (Dependence of $\hat{t}_{AR}$ on $\widehat{t}$, $\widehat{\widehat{\rho}}$, and $\widehat{F}$)**.**

$$\hat{t}_{AR}^2 = \frac{\hat{t}^2}{1 + 2\widehat{\widehat{\rho}}\frac{\hat{t}}{\sqrt{\widehat{F}}} + \frac{\hat{t}^2}{\widehat{F}}}$$

where

$$\widehat{\widehat{\rho}} = \frac{\widehat{C}(Z\widehat{u}, Z\widehat{v})}{\sqrt{\widehat{V}(Z\widehat{u})}\sqrt{\widehat{V}(Z\widehat{v})}}$$

PROOF: The proof is similar to Lemma 4, but with some subtle differences. First, note that since $\widehat{u}_0 = \widehat{u} + (\widehat{\beta}-\beta_0)\widehat{v}$, we can write

$$\frac{\hat{t}_{AR}^2(\beta_0)}{\hat{t}^2} = \frac{\widehat{V}(Z\widehat{u})}{\widehat{V}(Z\widehat{u}_0)} = \frac{\widehat{V}(Z\widehat{u})}{\widehat{V}(Z\widehat{u}) + 2(\widehat{\beta}-\beta_0)\widehat{C}(Z\widehat{u}, Z\widehat{v}) + (\widehat{\beta}-\beta_0)^2\widehat{V}(Z\widehat{v})}$$

The result follows after dividing top and bottom by $\widehat{V}(Z\widehat{u})$ and recognizing that

$$\frac{\hat{t}}{\sqrt{\widehat{F}}} = \left(\widehat{\beta}-\beta_0\right)\sqrt{\frac{\widehat{V}(Z\widehat{v})}{\widehat{V}(Z\widehat{u})}}$$

Note that unlike Lemma 4, the dependence is (1) on $\widehat{F}$ as opposed to $\widehat{f}$ and (2) on the (generalized) correlation between the *IV* residual and the first-stage residual, as opposed to the (generalized) correlation between the *AR* residual and the first-stage residual. ⊠

A.6 Rejection probabilities for tests based on t-ratio

We next derive an asymptotic version of Lemma 3.

Lemma 6 (Limiting Distribution of $\widehat{t}^2$ Under Weak IV Asymptotics)**.** *Under Assumptions 1, 2, and 3, we have*

$$\widehat{t}^2 \quad \xrightarrow{d} \quad \frac{t_{AR}^2(\beta_0)}{1-2\rho(\beta_0)\frac{t_{AR}(\beta_0)}{f}+\frac{t_{AR}^2(\beta_0)}{f^2}} \equiv t^2(\beta_0)$$

where

$$\rho(\beta_0) \quad = \quad \lim_{N\to\infty}\frac{C\left(\frac{1}{N}\mathbf{Z}'\mathbf{u_0},\frac{1}{N}\mathbf{Z}'\mathbf{v}\right)}{\sqrt{V\left(\frac{1}{N}\mathbf{Z}'\mathbf{u_0}\right)}\sqrt{V\left(\frac{1}{N}\mathbf{Z}'\mathbf{v}\right)}}$$

and $t_{AR}(\beta_0)$ and f are distributed jointly normal with unit variances, correlation $\rho(\beta_0)$, and means that are given below.

PROOF: We will show that regardless of whether β_0 is the true parameter or not,

$$\begin{pmatrix}\widehat{t}_{AR}(\beta_0)\\ \widehat{f}\end{pmatrix} \quad \xrightarrow{d} \quad \begin{pmatrix}t_{AR}(\beta_0)\\ f\end{pmatrix} \sim N\left(f_0\begin{pmatrix}\frac{\Delta(\beta_0)}{\sqrt{1+2\rho(\beta_0)\Delta(\beta_0)+\Delta^2(\beta_0)}}\\ 1\end{pmatrix},\begin{pmatrix}1 & \rho(\beta_0)\\ \rho(\beta_0) & 1\end{pmatrix}\right)$$

where

$$\begin{aligned}\Delta(\beta_0) &= (\beta-\beta_0)\frac{\sqrt{\sigma_v^2}}{\sqrt{\sigma_\varepsilon^2-2\beta\sigma_{\varepsilon v}+\beta^2\sigma_v^2}}\\ f_0 &= \frac{\pi_{ZZ}}{\sqrt{\sigma_v^2}}\end{aligned}$$

from which the result follows.

Since $\varepsilon = u+v\beta$, the *AR* outcome $u_0 = Y-X\beta_0$ can be written as

$$\begin{aligned}u_0 &= X\beta+u-X\beta_0 = (Z\pi+v)(\beta-\beta_0)+u = Z\pi(\beta-\beta_0)+v(\beta-\beta_0)+\varepsilon-v\beta\\ &= Z\pi(\beta-\beta_0)+\varepsilon-v\beta_0\end{aligned}$$

which means the *AR* coefficient is given by

$$\begin{aligned}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{u}_0 &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left[\mathbf{Z}'\pi_N(\beta-\beta_0)+\varepsilon-\mathbf{v}\beta_0\right] \\ &= \pi(\beta-\beta_0)+(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left(\varepsilon-\beta_0\mathbf{v}\right)\end{aligned}$$

The *AR* standard error is the square root of the estimated variance of the above, i.e.:

$$(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}\left((\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'(\varepsilon-\beta_0 v)\right)=(\mathbf{Z}'\mathbf{Z})^{-1}\left(\widehat{\sigma}_\varepsilon^2-2\beta_0\widehat{\sigma}_{\varepsilon v}+\beta_0^2\widehat{\sigma}_v^2\right)$$

and therefore the *AR* statistic is given by

$$\begin{aligned}\widehat{t}_{AR}(\beta_0) &= \frac{\pi_N(\beta-\beta_0)+(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left(\varepsilon-\beta_0 v\right)}{\sqrt{(\mathbf{Z}'\mathbf{Z})^{-1}\left(\widehat{\sigma}_\varepsilon^2-2\beta_0\widehat{\sigma}_{\varepsilon v}+\beta_0^2\widehat{\sigma}_v^2\right)}} \\ &= \frac{\pi_{ZZ}(\beta-\beta_0)}{\sqrt{\sigma_\varepsilon^2-2\beta_0\sigma_{\varepsilon v}+\beta_0^2\sigma_v^2}}+\frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'\left(\varepsilon-\beta_0 v\right)}{\sqrt{\widehat{\sigma}_\varepsilon^2-2\beta_0\widehat{\sigma}_{\varepsilon v}+\beta_0^2\widehat{\sigma}_v^2}}+o_p(1)\end{aligned}$$

Similarly,

$$\widehat{f} = \frac{(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}}{\sqrt{(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}(Z\widehat{v})(\mathbf{Z}'\mathbf{Z})^{-1}}}=\frac{(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{Z}'\pi_N+v)}{\sqrt{(\mathbf{Z}'\mathbf{Z})^{-1}\widehat{V}((\mathbf{Z}'\mathbf{Z})^{-1/2}Z\widehat{v})}}=\frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'(\mathbf{Z}'\pi_N+v)}{\sqrt{\widehat{\sigma}_v^2}}$$

Some algebra shows that

$$C\left(\widehat{t}_{AR}(\beta_0),\widehat{f}\right) = \frac{\sigma_{\varepsilon v}-\beta_0\sigma_v^2}{\sqrt{\sigma_v^2}\sqrt{\sigma_\varepsilon^2-2\beta_0\sigma_{\varepsilon v}+\beta_0^2\sigma_v^2}}+o_p(1)$$

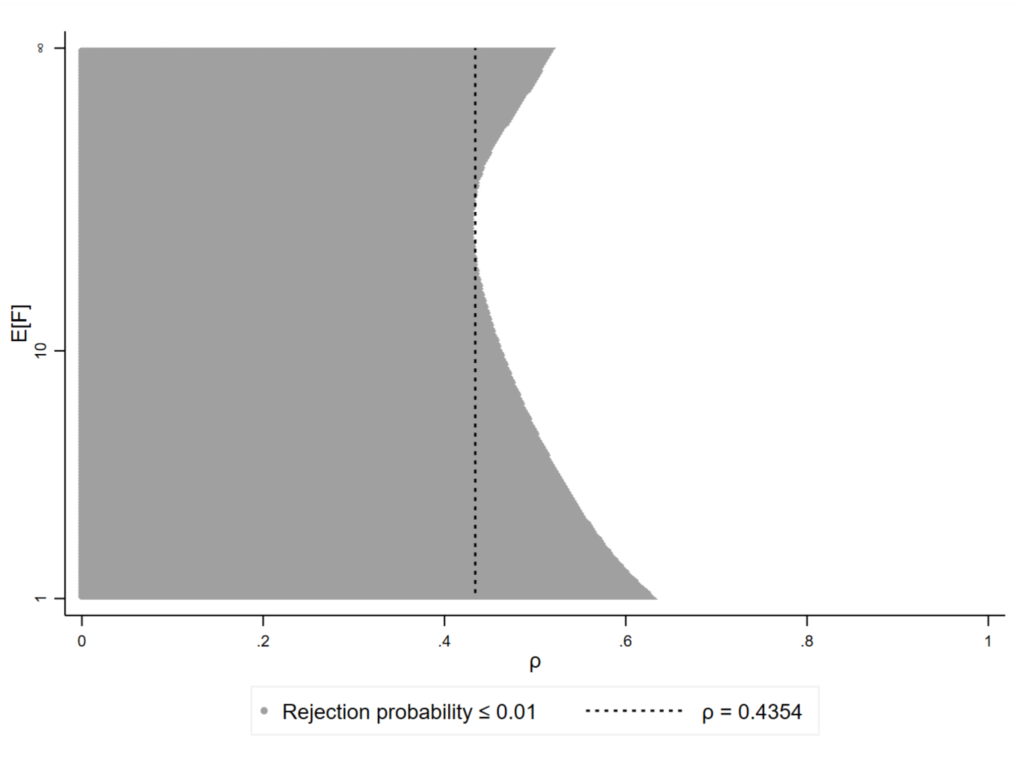
and that the first term in the above is equal to $\rho(\beta_0)$.

Putting these results together, we have

$$\begin{pmatrix}\widehat{t}_{AR}(\beta_0)-f_0\frac{\Delta(\beta_0)}{1+2\rho(\beta_0)\Delta(\beta_0)+\Delta^2(\beta_0)} \\ \widehat{f}-f_0\end{pmatrix}=\begin{pmatrix}\frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'(\varepsilon-\beta_0 v)}{\sqrt{\widehat{\sigma}_\varepsilon^2-2\beta_0\widehat{\sigma}_{\varepsilon v}+\beta_0^2\widehat{\sigma}_v^2}} \\ \frac{(\mathbf{Z}'\mathbf{Z})^{-1/2}\mathbf{Z}'\mathbf{v}}{\sqrt{\widehat{\sigma}_v^2}}\end{pmatrix}+o_p(1)$$

and joint asymptotic normality of $(\hat{t}_{AR}(\beta_0),\widehat{f})$ of the stated form follows from Assumption 3 and the continuous mapping theorem. ☒

Figure A1: Combinations of $E[F]$, ρ for $Pr[t^2 > 2.58^2] \leq 0.01$



Vertical axis scale uses the transformation $\frac{\frac{E[F]}{10}}{1+\frac{E[F]}{10}}$. Shaded region represents all combinations of $E[F], \rho$ such that the rejection probability is less than or equal to 0.01. Dashed line is the maximum ρ such that the region to the left is shaded.

A.7 Some numerical findings and other results derived from the rejection probabilities

Result 1a. *In addition to the IV model in (1), consider the restriction that* $E[F] \geq \bar{F}$. *The smallest value of* $\bar{F}$ *such that* $\Pr\left[t^2 > 1.96^2\right] \leq .05$ *is 142.6 .*

Result 1b. *In addition to the IV model in (1), consider the restriction that* $|\rho| < \bar{\rho}$. *The largest value of* $\bar{\rho}$ *such that* $\Pr\left[t^2 > 1.96^2\right] \leq .05$ *is 0.565.*

Result 1c. *For the 1 percent level of significance, there exists no* $\bar{F}$ *such that* $\Pr\left[t^2 > 2.58^2\right] \leq 0.01$ *for all* $E[F] \geq \bar{F}$*, and the largest* $\bar{\rho}$ *such that* $\Pr\left[t^2 > 2.58^2\right] \leq 0.01$ *for all* $|\rho| \leq \bar{\rho}$ *is 0.43. The full set of values of* $|\rho|, E[F]$ *for which* $\Pr\left[t^2 > 2.58^2\right] \leq 0.01$ *is illustrated in Figure A1.*

Result 2a. $\Pr\left[\left\{t^2 > 1.96^2\right\} \cap \{F > 10\}\right] \leq 0.113$ *for all values of* $\rho, E[F]$. *This implies that confidence intervals are* $\hat{\beta}_{IV} \pm 1.96 \cdot \hat{SE}\left(\hat{\beta}_{IV}\right)$ *when* $F \geq 10$ *and* $(-\infty, \infty)$ *when* $F < 10$*, and should be interpreted as 88.7 percent confidence intervals.*

Result 2b. $\Pr\left[\left\{t^2 > 1.96^2\right\} \cap \{F \geq 104.7\}\right] \leq 0.05$ *for all values of* $\rho, E[F]$.

Result 2c. $\Pr\left[\left\{t^2 > 3.43^2\right\} \cap \{F > 10\}\right] \leq 0.05$ *for all values of* $\rho, E[F]$.

Result 2d. *Let AR be the statistic of There exists no finite threshold* $\bar{F}$ *such that* $\Pr\left[\left\{t^2 > 1.96^2\right\} \cap \{F \geq \bar{F}\}\right] + \Pr\left[\left\{AR > 1.96^2\right\} \cap \{F < \bar{F}\}\right] \leq 0.05$ *for all values of* $\rho, E[F]$.

Derivation of Results 1a-b-c, 2a-b-c-d:

Recall

$$t^2(f, t_{AR}) = \frac{f^2 t_{AR}^2}{f^2 - 2\rho_0 f t_{AR} + t_{AR}^2}$$

Lemma 7. *For* $\rho_0 = \pm 1$*, suppose* $f = f_0^* + \rho_0 t_{AR}$. *Then, for* $q > 0$*,*

$$\{t_{AR} : t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) \geq q\} = \begin{cases} (-\infty, \underline{f}_A^*] \cup [\bar{f}_A^*, \infty) & \text{if } |f_0^*| < 4\sqrt{q} \\ (-\infty, \underline{f}_A^*] \cup \{-\frac{\rho_0 f_0^*}{2}\} \cup [\bar{f}_A^*, \infty) & \text{if } |f_0^*| = 4\sqrt{q} \\ (-\infty, \underline{f}_A^*] \cup [\underline{f}_B^*, \bar{f}_B^*] \cup [\bar{f}_A^*, \infty) & \text{if } |f_0^*| > 4\sqrt{q} \end{cases}$$

where

$$\underline{f}_A^* = \frac{-\rho_0 f_0^* - \sqrt{f_0^{*2} + 4|f_0^*|\sqrt{q}}}{2}; \qquad \bar{f}_A^* = \frac{-\rho_0 f_0^* + \sqrt{f_0^{*2} + 4|f_0^*|\sqrt{q}}}{2}$$

$$\underline{f}_B^* = \frac{-\rho_0 f_0^* - \sqrt{f_0^{*2} - 4|f_0^*|\sqrt{q}}}{2}; \qquad \bar{f}_B^* = \frac{-\rho_0 f_0^* + \sqrt{f_0^{*2} - 4|f_0^*|\sqrt{q}}}{2}$$

PROOF:

$$t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) = \frac{1}{f_0^{*2}}(\rho_0 f_0^* + t_{AR})^2 t_{AR}^2$$

Let $\underline{\tau} = \min\{-\rho_0 f_0^*, 0\}$ and $\bar{\tau} = \max\{-\rho_0 f_0^*, 0\}$. Note $t^2(f_0^* + \rho_0 t_{AR}, t_{AR})$ is a quartic polynomial, monotonically decreasing on $(-\infty, \underline{\tau})$ and $(-\frac{\rho_0 f_0^*}{2}, \bar{\tau})$ and monotonically increasing on $(\underline{\tau}, -\frac{f_0}{2})$ and $(\bar{\tau}, \infty)$. So the solutions to $t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) = q$

are as follows:

$$\{t_{AR}: t^2(f_0^* + \rho_0 t_{AR}, t_{AR}) = q\} \; = \; \begin{cases} \{\underline{f}_A^*, \bar{f}_A^*\} & \text{if } |f_0^*| < 4\sqrt{q} \\ \{\underline{f}_A^*, \bar{f}_A^*, -\frac{\rho_0 f_0^*}{2}\} & \text{if } |f_0^*| = 4\sqrt{q} \\ \{\underline{f}_A^*, \bar{f}_A^*, \underline{f}_B^*, \bar{f}_B^*\} & \text{if } |f_0^*| > 4\sqrt{q} \end{cases}$$

The result follows. ⊠

Remarks:

1. This result characterizes the rejection region for Wald when $\rho_0 = \pm 1$ under the null and alternative.
2. Our asymptotic approximation is based on: $\begin{pmatrix} t_{AR} \\ f \end{pmatrix} \sim N\left(\begin{pmatrix} t_1 \\ f_0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{pmatrix} \right)$ When $\rho_0 = \pm 1$, $f = f_0 - \rho_0 t_1 + \rho_0 t_{AR}$. So, Lemma 7 can be used to characterize the corresponding Wald rejection region with $f_0^* = f_0 - \rho_0 t_1$. Note that under the null, $t_1 = 0$ and $f_0^* = f_0$.
3. Under the null, $f_0^* = f_0$, so define

$$\underline{f}_A = \frac{-\rho_0 f_0 - \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{2}; \qquad \bar{f}_A = \frac{-\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{2}$$

$$\underline{f}_B = \frac{-\rho_0 f_0 - \sqrt{f_0^2 - 4|f_0|\sqrt{q}}}{2}; \qquad \bar{f}_B = \frac{-\rho_0 f_0 + \sqrt{f_0^2 - 4|f_0|\sqrt{q}}}{2}$$

   Then,

$$\Pr_{f_0, \rho_0 = \pm 1}(t^2 \geq q) = \begin{cases} \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) & \text{if } |f_0| \leq 4\sqrt{q} \\ \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B) & \text{if } |f_0| > 4\sqrt{q} \end{cases}$$

   where Φ denotes the standard normal c.d.f.
4. This result can also be used to characterize $\{t_{AR}: t^2 \geq q, f^2 \geq \bar{F}\}$ by intersecting the set given with $(-\infty, -\sqrt{\bar{F}} - \rho_0 f_0^*] \cup [\sqrt{\bar{F}} - \rho_0 f_0^*, \infty)$.

Corollary 1. *Under the null,*

(a) $\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = \Pr_{-f_0,\rho_0=-1}(t^2 \geq q, f^2 \geq \bar{F}) = \Pr_{-f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$

(b) $\lim_{f_0 \downarrow 0} \Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1 - [\Phi(\sqrt{\bar{F}}) - \Phi(-\sqrt{\bar{F}})]$

(c) $\lim_{f_0 \longrightarrow \infty} \Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$

PROOF:
(a) Subscripting t with ρ_0 to denote its direct dependence on ρ_0, note that $t^2_{\rho_0}(f_0^* + \rho_0 t_{AR}, t_{AR}) = t^2_{-\rho_0}(-(f_0^* + \rho_0 t_{AR}), t_{AR}) = t^2_{-\rho_0}(-f_0^* + (-\rho_0)t_{AR}), t_{AR})$ and $f^2 = (f_0^* + \rho_0 t_{AR})^2 = (-f_0^* + (-\rho_0)t_{AR})^2$. The first equality follows.
Next, $t^2_{\rho_0}(f_0^* + \rho_0 t_{AR}, t_{AR}) = t^2_{\rho_0}(-f_0^* + \rho_0(-t_{AR}), (-t_{AR}))$ and $f^2 = (f_0^* + \rho_0 t_{AR})^2 = (-f_0^* + \rho_0(-t_{AR}))^2$. Under the null, $t_1 = 0$ and $t_{AR} \sim N(0,1)$ is symmetrically distributed about zero. The second equality follows.

(b) Note that $\underline{f}_A$, $\bar{f}_A \longrightarrow 0$ as $f_0 \longrightarrow 0$. The result follows.

(c)

$$\bar{f}_A = \frac{-\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{2} \left(\frac{\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}}{\rho_0 f_0 + \sqrt{f_0^2 + 4|f_0|\sqrt{q}}} \right) = \frac{2\sqrt{q}}{\rho_0 \frac{f_0}{|f_0|} + \sqrt{1 + \frac{4\sqrt{q}}{|f_0|}}}$$

Hence, $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \bar{f}_A = \sqrt{q}$. Similarly, $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \underline{f}_A = -\infty$; $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \bar{f}_B = -\sqrt{q}$; $\lim_{\rho_0=1, f_0 \longrightarrow \infty} \underline{f}_B = -\infty$. When $\rho_0 = 1$, as $f_0 \longrightarrow \infty$, $\sqrt{\bar{F}} - f_0 \longrightarrow -\infty$, so that the rejection probability is determined by $\bar{f}_A$ and $\bar{f}_B$ asymptotically. Result (c) follows. ⊠

Remarks:

1. Note that results on rejection probabilities for Wald follow setting $\bar{F} = 0$, $\Pr_{f_0,\rho_0}(t^2 \geq q) = \Pr_{f_0,\rho_0}(t^2 \geq q, f^2 \geq 0)$.
2. By part (a), under the null, to characterize $\Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q, f^2 \geq \bar{F})$, it suffices to focus on the case $\rho_0 = 1$ and $f_0 \geq 0$.
3. From (b), by choosing $\bar{F}$ close to zero, the worst case rejection probability for $\{t^2 \geq q, f^2 \geq \bar{F}\}$ is arbitrarily close to one.
4. By parts (a) and (b), $\lim_{f_0 \longrightarrow 0} \Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q) = 1$

Corollary 2. *Under the null, there exists $\bar{f}_0 > 0$ large enough that for any $f_0 > \bar{f}_0$,*

(a) if $q < 4$, then
$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) > 0;$$

(b) if $q > 4$, then
$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0.$$

PROOF:
Set $\rho_0 = 1$. As $f_0 \longrightarrow \infty$, $\Pr_{f_0,\rho_0=1}(t^2 \geq q) = \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)$. Define $v = \frac{1}{f_0}$. So $v \downarrow 0$ as $f_0 \longrightarrow \infty$. From the Proof of Lemma 1, for $f_0 > 0$, we have $\bar{f}_A = \frac{2\sqrt{q}}{1+\sqrt{1+4v\sqrt{q}}}$. Similarly, $\bar{f}_B = \frac{-2\sqrt{q}}{1+\sqrt{1-4v\sqrt{q}}}$.

$$\frac{\partial \bar{f}_A}{\partial v} = \frac{-4q}{(1+\sqrt{1+4v\sqrt{q}})^2\sqrt{1+4v\sqrt{q}}}\;; \quad \frac{\partial \bar{f}_B}{\partial v} = \frac{-4q}{(1+\sqrt{1-4v\sqrt{q}})^2\sqrt{1-4v\sqrt{q}}}$$

Let $w = 4v\sqrt{q}$.

$$\begin{aligned}
\frac{\partial}{\partial v}\left[1-\Phi(\bar{f}_A)+\Phi(\bar{f}_B)\right] &= \phi(\bar{f}_B)\frac{\partial \bar{f}_B}{\partial v} - \phi(\bar{f}_A)\frac{\partial \bar{f}_A}{\partial v} \\
&= \phi(\bar{f}_B)\frac{\partial \bar{f}_B}{\partial v}\left[1-\frac{(1+\sqrt{1-4v\sqrt{q}})^2\sqrt{1-4v\sqrt{q}}}{(1+\sqrt{1+4v\sqrt{q}})^2\sqrt{1+4v\sqrt{q}}}\frac{\phi(\bar{f}_A)}{\phi(\bar{f}_B)}\right] \\
&= -\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right|\left[1-\frac{(1+\sqrt{1-w})^2\sqrt{1-w}}{(1+\sqrt{1+w})^2\sqrt{1+w}}\exp\left(2q\left[\frac{-1}{(1+\sqrt{1+w})^2}+\frac{1}{(1+\sqrt{1-w})^2}\right]\right)\right]
\end{aligned}$$

Using a first-order expansion of the bracketed term in the final expression above, we find that as $w \downarrow 0$,

$$\begin{aligned}
\frac{\partial}{\partial v}\left[1-\Phi(\bar{f}_A)+\Phi(\bar{f}_B)\right] &= -\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right|\left[(4-q)2v\sqrt{q}+o(v)\right] \\
&= \left[(q-4)2v\sqrt{q}+o(v)\right]\cdot\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right|
\end{aligned}$$

Notice from the Proof of Lemma 1, $\lim_{\rho_0=1,f_0\longrightarrow\infty}\underline{f}_A = \lim_{\rho_0=1,f_0\longrightarrow\infty}\underline{f}_B = -\infty$. Correspondingly, it is straightforward to show that the terms $\Phi(\underline{f}_A)$ and $\Phi(\underline{f}_B)$ do not have a first-order effect on the derivative above (for cases $\rho_0 = 1$ and $f_0 \longrightarrow \infty$, or $\rho_0 = -1$ and $f_0 \longrightarrow -\infty$).

So, under the null, for $q \neq 4$, as $f_0 \longrightarrow \infty$,

$$\begin{aligned}
\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q) &= \frac{\partial}{\partial f_0}\left[\Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)\right] \\
&= -\frac{1}{f_0^2}\frac{\partial}{\partial v}\left[\Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)\right] \\
&= -v^2\frac{\partial}{\partial v}\left[\Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + \Phi(\bar{f}_B) - \Phi(\underline{f}_B)\right] \\
&= (4-q)\cdot\underbrace{\left[2\sqrt{q}\phi(\bar{f}_B)\left|\frac{\partial \bar{f}_B}{\partial v}\right| v^3\right]}_{>0} + o(v^3)
\end{aligned}$$

This established the result for the Wald rejection region. The generalization to $\{t^2 \geq q, f^2 \geq \bar{F}\}$ is straightforward and follows the argument above, as $\Phi(\bar{f}_A)$ and $\Phi(\bar{f}_B)$ are still the dominant terms in the derivative. ⊠

Remarks:

1. Putting Corollary 1(c) and Corollary 2 together, we see that the rejection probability for Wald with $\rho_0 = 1$ asymptotes to $1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$ as $f_0 \longrightarrow \infty$. When $q < 4$, the Wald rejection probability approaches its asymptote from below. This means that for large enough f_0, $\Pr_{f_0,\rho_0=1}(t^2 \geq q) < 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$. Given Corollary 1(a) and continuity of the Wald rejection probability, there exists a value f_0 such that $\Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q) = 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$.

2. When $q > 4$, the rejection probability for Wald with $\rho_0 = \pm 1$ is decreasing as it asymptotes to $1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$. Generally, there will *not* be a value of f_0 such that $\Pr_{f_0,\rho_0=\pm 1}(t^2 \geq q) = 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})]$.

3. $q = 4$ corresponds to test size 4.55%. So, $q < 4$ corresponds to test size > 4.55%, and $q > 4$ corresponds to test size < 4.55%.

Derivation of Result 1a: We use numerical evidence to verify that for a given $f_0 > 0$, the largest null rejection probability occurs with $\rho_0 = 1$. As discussed in Remark 1 above, taking $q = 1.96^2 < 4$, Corollary 2(a) and Corollary 1(c) then tell us that there exists f_0 such that $\Pr_{f_0,\rho_0=1}(t^2 \geq q) < 1 - [\Phi(\sqrt{q}) - \Phi(-\sqrt{q})] = .05$. From Lemma 7, we have $\Pr_{f_0,\rho_0=1}(t^2 \geq q) = \Phi(\underline{f}_A) + 1 - \Phi(\bar{f}_A) + [\Phi(\bar{f}_B) - \Phi(\underline{f}_B)]\mathbf{1}\{|f_0| > 4\sqrt{q}\}$. Given the formulas for $\underline{f}_A$, $\bar{f}_A$, $\underline{f}_B$, and $\bar{f}_B$ above, it is straightforward to solve for the smallest f_0 such that $\Pr_{f_0,\rho_0=1}(t^2 \geq q) = .05$ and verify that

$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q) > 0$ for any larger f_0 (so that $\Pr_{f_0,\rho_0=1}(t^2 \geq q)$ must be smaller than its asymptotic value of .05 for all larger f_0). The solution is $f_0 = 11.9$. Hence $E(F) = E(f^2) = Var(f) + [E(f)]^2 = 1 + (11.9)^2 = 142.6$. ⊠

Derivation of Results 1b and 1c: Taking $q = 2.58^2 > 4$, Corollary 2(b) says that for large enough f_0, $\Pr_{f_0,\rho_0=1}(t^2 \geq q) > .01$. We verify that the derivative $\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q) < 0$ for large enough f_0 and then verify the inequality numerically for any smaller values of f_0. The findings in Results 1b and 1c for $\rho_0 < 1$ are obtained numerically.

⊠

Define $\phi^* = \frac{\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}$, and $\bar{\phi} = \begin{cases} 4\sqrt{q} & \text{if } \bar{F} \leq 4\sqrt{q} \\ \frac{\bar{F}}{\sqrt{\bar{F}}-\sqrt{q}} & \text{if } \bar{F} > 4q \end{cases}$

Lemma 8. *Under the null, for $\bar{F} > 0$,*

if $0 < f_0 < \phi^*$,

$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) > 0\,;$$

and if $\phi^* < f_0 < \bar{\phi}$,

$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0.$$

PROOF: For $0 < f_0 < \phi^*$, $-f_0 + \sqrt{\bar{F}} > \bar{f}_A(f_0)$, and for $f_0 > \phi^*$, $-f_0 + \sqrt{\bar{F}} < \bar{f}_A(f_0)$. Let $\underline{\phi} = \begin{cases} \frac{\bar{F}}{\sqrt{q}-\sqrt{\bar{F}}} & \bar{F} < q \\ \infty & \bar{F} \geq q \end{cases}$. If $0 < f_0 < \underline{\phi}$, then $-f_0 - \sqrt{\bar{F}} < \underline{f}_A(f_0)$. Moreover, $\frac{\partial}{\partial f_0}[-f_0 - \sqrt{\bar{F}}] < 0$ and $\frac{\partial}{\partial f_0}\underline{f}_A(f_0) < 0$ for $f_0 > 0$. For $0 < f_0 < \bar{\phi}$, we can show that $[\underline{f}_B(f_0), \bar{f}_B(f_0)] \cap \left((-\infty, -f_0 - \sqrt{\bar{F}}] \cup [-f_0 + \sqrt{\bar{F}}, \infty)\right) = \emptyset$. Hence,

$$\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = \begin{cases} 1 - \Phi(-f_0 + \sqrt{\bar{F}}) + \Phi(-f_0 - \sqrt{\bar{F}}) & \text{if } 0 < f_0 < \phi^* \\ 1 - \Phi(\bar{f}_A(f_0)) + \Phi(-f_0 - \sqrt{\bar{F}}) & \text{if } \phi^* < f_0 < \underline{\phi} \end{cases}.$$

For $0 < f_0 < \phi^*$,

$$\frac{\partial}{\partial f_0}\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = \phi(-f_0 + \sqrt{\bar{F}}) - \phi(-f_0 - \sqrt{\bar{F}}) > 0$$

since $|-f_0+\sqrt{\bar{F}}| < |-f_0-\sqrt{\bar{F}}|$. And, for $\phi^* < f_0 < \bar{\phi}$, $\frac{\partial}{\partial f_0}[1-\Phi(\bar{f}_A(f_0))+\Phi(-f_0-\sqrt{\bar{F}})] < 0$ and $\frac{\partial}{\partial f_0}[1-\Phi(\bar{f}_A(f_0))+\Phi(\underline{f}_A(f_0))] < 0$. The result follows. ⊠

Remarks:

1. Lemma 8 characterizes a local maximum in $\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$. The maximum occurs at $f_0 = \phi^*$, which is the smallest maximizing point for $f_0 > 0$.

2. Importantly, note that the derivative of $\Pr_{f_0,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$ is discontinuous at $f_0 = \phi^*$, so this maximizer is well separated, which is useful for our numerical analysis.

3. We know the asymptotic value of this rejection probability by Corollary 1(c). In addition, numerical experimentation shows another bounded local maximum can sometimes be the global maximizer when $q > 4$, as might be expected given Corollary 2.

4. $$\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right) \tag{6}$$

   - $\frac{\partial}{\partial \bar{F}}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0$ and $\frac{\partial}{\partial q}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 0$
   - $\lim_{\bar{F}\downarrow 0}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1$
   - $\lim_{\bar{F}\longrightarrow\infty}\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) = 1-\Phi(\sqrt{q})$.
   - Clearly, $\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$ cannot be a global maximizer over $f_0 > 0$ if $\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F}) < 1-[\Phi(\sqrt{q})-\Phi(-\sqrt{q})]$

Size Calculations

Equation (6) is a key step in our size calculation results. We use Lemma 8 and numerical evidence to verify that $\Pr_{f_0=\phi^*,\rho_0=1}(t^2 \geq q, f^2 \geq \bar{F})$ is a global maximizer over f_0, ρ_0. To achieve a size γ test, we solve

$$\gamma = 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right)$$

Note that the expression on the righthand side is monotonic decreasing in both $\bar{F}$ and q, so that solving this equation for $\bar{F}$ or q is straightforward.

Derivation of Result 2a: Set $\bar{F} = 10$ and $q = 1.96^2$. Then, $\gamma = 0.113$.[35] $\boxtimes$

Derivation of Result 2b: Set $q = 1.96^2$ and $\gamma = .05$. Then, solve for $\bar{F}$ yielding $\bar{F} = 104.7$. $\boxtimes$

Derivation of Result 2c: Set $\bar{F} = 10$ and $\gamma = .05$. Then, solve for q yielding $q = 3.4$. $\boxtimes$

Derivation of Result 2d: Let $f_0 = \phi^*$. Then,

$$\Pr_{f_0=\phi^*,\rho_0=1}(\{t^2 \geq q, f^2 \geq \bar{F}\} \cup \{t_{AR}^2 \geq q, f^2 < \bar{F}\}) = \begin{cases} 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right) \\ \text{if } \bar{F} \leq \frac{q}{2} \\ 1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi(-\sqrt{q}) \\ \text{if } \bar{F} > \frac{q}{2} \end{cases}$$

Note that $1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi(-\sqrt{q}) > 1-\Phi(\sqrt{q})+\Phi(-\sqrt{q})$. When $\bar{F} = \frac{q}{2}$, the expressions in the bracket above are equal. Since we already know $1-\Phi\left(\frac{\sqrt{\bar{F}q}}{\sqrt{\bar{F}}+\sqrt{q}}\right)+\Phi\left(\frac{-\sqrt{\bar{F}q}-2\bar{F}}{\sqrt{\bar{F}}+\sqrt{q}}\right)$ is decreasing in $\bar{F}$, we can conclude that for all $\bar{F}$,

$$\Pr_{f_0=\phi^*,\rho_0=1}(\{t^2 \geq q, f^2 \geq \bar{F}\} \cup \{t_{AR}^2 \geq q, f^2 < \bar{F}\}) > 1-\Phi(\sqrt{q})+\Phi(-\sqrt{q}).$$

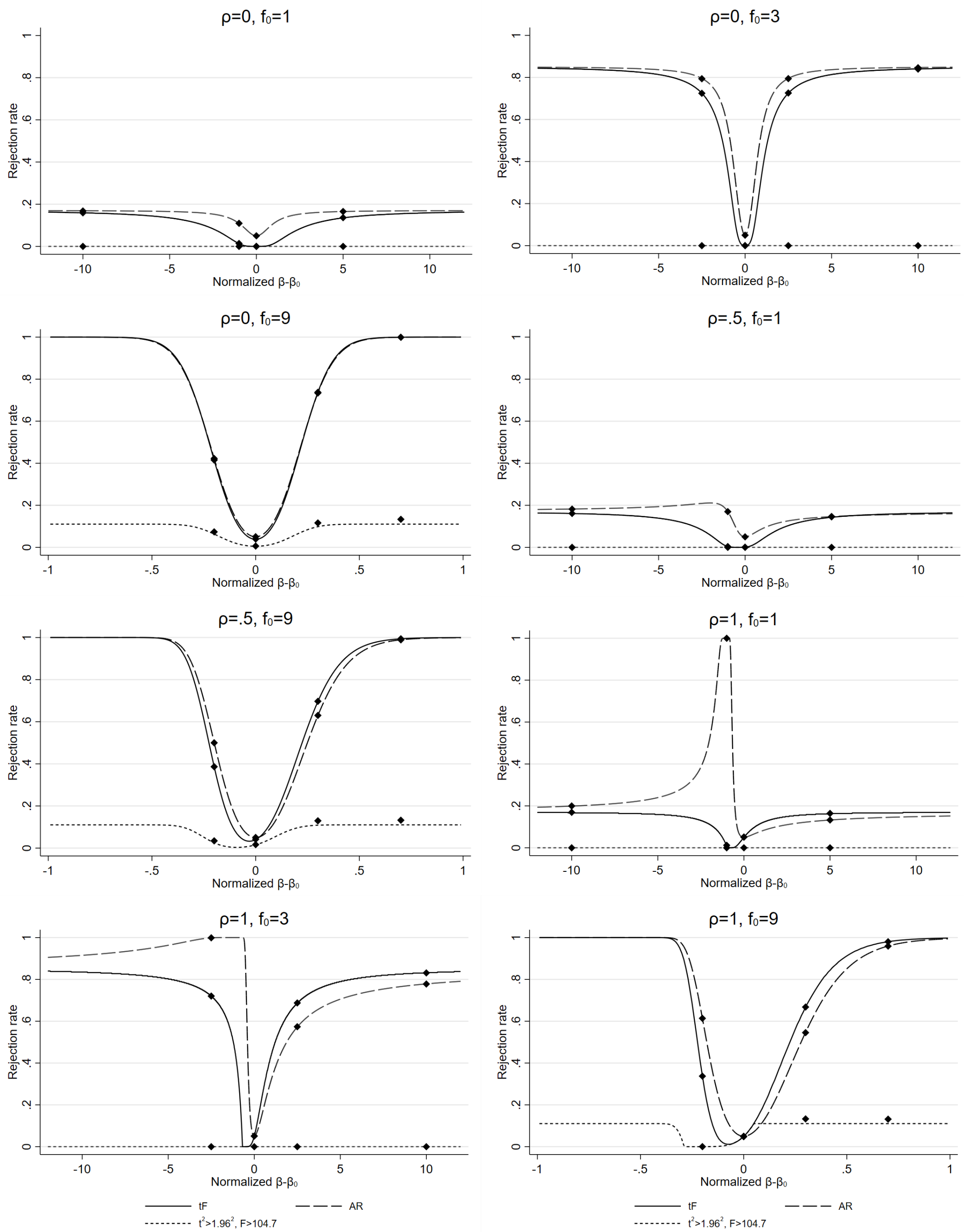
Plugging in $q = 1.96^2$ yields the stated result. $\boxtimes$

A.8 Power curves: AR, tF, and step functions (c^*, F^*)

Figure A2 contains the power curves for the eight remaining scenarios as described in the text. A black diamond represents the rejection probability from 250,000 Monte Carlo simulations, each with a sample size of 1,000.

[35] To be precise, we set q to the 95% quantile of the χ_1^2 distribution.

Figure A2: Power Curves



B The tF critical value function: Existence, Uniqueness, Size Control

B.1 Existence and Uniqueness

Define $\underline{d} = \left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2$, $b = 3\underline{d} - \frac{d^2}{2} + \frac{d^3}{6}$, and consider the functional equation in F and $c_\alpha(\cdot)$

$$c_\alpha\left(\left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}\right]^2\right) \tag{7}$$

$$= \frac{\left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}\right]^2}{\left[\frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}\right]^2}.$$

$$\cdot\left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}\right) - (1-\alpha)\right)\right]^2$$

Lemma 9. *There exists a function $c_\alpha(\cdot)$ satisfying (7) for $F \in (\underline{d}, \underline{d}+\delta]$ for some $\delta > 0$ with the following properties:*

(i) $c_\alpha(F) - \left(\frac{d^3}{F-\underline{d}} - b\right) = O(\sqrt{F-\underline{d}})$ *as* $F \downarrow \underline{d}$

(ii) Let $\breve{c}_\alpha$ satisfy (7) for $F \in (\underline{d}, \underline{d}+\breve{\delta}]$ for some $\breve{\delta} > 0$ with $\breve{c}_\alpha(F) = \frac{d^3}{F-d} - b + o((F-d)^{-1/3})$ as $F \downarrow \underline{d}$. Then, $c_\alpha(F) = \breve{c}_\alpha(F)$ for $F \in (\underline{d}, \underline{d}+\delta_1]$ and some $\delta_1 > 0$;

(iii) $c_\alpha \in C^\infty$ *on* $(\underline{d}, \underline{d}+\delta]$;

(iv) For any $k > 0$, there exists $\delta_2 > 0$ such that $c_\alpha(F) \geq k$ for $F \in (\underline{d}, \underline{d}+\delta_2]$, and $c_\alpha(F)$ is decreasing for $F \in (\underline{d}, \underline{d}+\delta_3]$ for some $\delta_3 > 0$.

PROOF: To show the desired existence, we will transform equation (7) to put it into canonical form for results from the dynamical systems literature. Once in canonical form, we find that (7) is a degenerate case to which the standard stable manifold theorem does not apply. New results from Fefferman (2021) obtained for our case provide the desired existence and uniqueness.

Based on (7), define the map $T : (F,y) \mapsto (\nu,\eta)$ where

$$\nu(F,y) = \left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{y}+\sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{y}+\sqrt{F}}\right]^2 \tag{8}$$

$$\eta(F,y) = \frac{\left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{y}+\sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{y}+\sqrt{F}}\right]^2 \left[\Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{y}+\sqrt{F}}\right) - (1-\alpha)\right)\right]^2}{\left[\frac{F}{\sqrt{y}+\sqrt{F}}\right]^2}.$$

We will show existence of an invariant curve for the map T. In particular, a function c_α exists such that $T(\Pi) \subset \Pi$ where $\Pi = \{(F, c_\alpha(F)) \,|\, F \in (\underline{d}, \underline{d}+\delta]\}$ for some $\delta > 0$. Since $T(\Pi) \subset \Pi$,

$$\eta(F, c_\alpha(F)) = c_\alpha(\nu(F, c_\alpha(F))) \tag{9}$$

for all $F \in (\underline{d}, \underline{d}+\delta]$. Given the definitions of ν and η, (9) is exactly (8), so existence of the invariant curve for T yields a function $c_\alpha(\cdot)$ satisfying (7). We now turn to obtaining the desired invariant curve for T.

We will transform T to obtain an equivalent map with an approximation in canonical form. Let $c^*_\alpha(F) = (F - \underline{d})c_\alpha(F)$.

$$h(t,z) = \frac{(t+\underline{d})\sqrt{t}}{\sqrt{z}+\sqrt{t(t+\underline{d})}}$$

$$gh(t,z) = \Phi^{-1}\left(\Phi\left(\sqrt{t+\underline{d}} - \frac{(t+\underline{d})\sqrt{t}}{\sqrt{z}+\sqrt{t(t+\underline{d})}}\right) - (1-\alpha)\right)$$

$$g(t,z) = gh(t,z) + h(t,z)$$

$$\xi(F,z) = [g(F-\underline{d},z)]^2$$

$$\zeta(F,z) = \frac{\left([g(F-\underline{d},z)]^2 - \underline{d}\right)[g(F-\underline{d},z)]^2 [gh(F-\underline{d},z)]^2}{[h(F-\underline{d},z)]^2}$$

These functions define a dynamical system iterative map: $T^* : (F,z) \mapsto (\xi,\zeta)$ with a fixed point at $(\underline{d}, \underline{d}^3)$. Taking standard expansions in t and Lagrange remainders, we obtain

$$
\begin{aligned}
h(t,z) &= \frac{\underline{d}}{\sqrt{z}}\sqrt{t} - \frac{\underline{d}^{3/2}}{z}t + \left[\frac{1}{\sqrt{z}} + \frac{\underline{d}^2}{z^{3/2}}\right]t^{3/2} - \left[\frac{3\sqrt{\underline{d}}}{2z} + \frac{\underline{d}^{5/2}}{z^2}\right]t^2 + r_h(t,z)t^{5/2} \\
gh(t,z) &= -\sqrt{\underline{d}} - \frac{\underline{d}}{\sqrt{z}}\sqrt{t} + \left[\frac{1}{2\sqrt{\underline{d}}} + \frac{\underline{d}^{3/2}}{z} - \frac{\underline{d}^{5/2}}{z}\right]t + \left[\frac{\underline{d}-1}{\sqrt{z}} - \frac{\underline{d}^2(\underline{d}-1)^2}{z^{3/2}}\right]t^{3/2} \\
&\quad + \left[-\frac{1}{4\sqrt{\underline{d}}} - \frac{1}{8\underline{d}^{3/2}} + \frac{3\sqrt{\underline{d}}}{2z}(\underline{d}-1)^2 + \frac{\underline{d}^{5/2}-3\underline{d}^{7/2}}{z^2} + \frac{11\underline{d}^{9/2}}{4z^2} - \frac{13\underline{d}^{11/2}}{12z^2}\right]t^2 \\
&\quad + r_{gh}(t,z)t^{5/2}
\end{aligned}
$$

where the remainder terms $r_h(t,z)$ and $r_{gh}(t,z)$ can be bounded for t in a non-negative neighborhood of zero and z in a neighborhood of $\underline{d}^3$.

Corresponding expansions for $gh(t,z)$, $\xi(F,z)$, and $\zeta(F,z)$ follow. Re-centering the fixed point to the origin by the change of variables $\tau = \xi - \underline{d}$, $(t = F - \underline{d})$, $\mu = \zeta - \underline{d}^3$, $u = z - \underline{d}^3$, and then expanding in u in a neighborhood of zero yields:

$$
\begin{aligned}
\tau(t,u) &= t - \frac{4}{\underline{d}}t^{3/2} - \frac{2}{\underline{d}^3}ut + R_\tau \\
\mu(t,u) &= -u + \left[-6\underline{d} + \underline{d}^2 - \frac{\underline{d}^3}{3}\right]t - \frac{2(2+\underline{d})}{\underline{d}}u\sqrt{t} + O(|(\sqrt{t},u)|^3)
\end{aligned}
$$

where $R_\tau = \sum_{\iota=2}^{4}\tilde{r}_\iota(t,u)(\sqrt{t})^\iota u^{4-\iota}$ and the remainder terms $\tilde{r}_\iota(t,u)$ can be bounded for t in a non-negative neighborhood of zero and u in a neighborhood of zero. The form of the remainder R_τ allows t to be factored out in τ:

$$
\tau = t\left[1 - \frac{4}{\underline{d}}t^{1/2} - \frac{2}{\underline{d}^3}u + \left(\sum_{\iota=0}^{2}\tilde{r}_\iota(t,u)(\sqrt{t})^\iota u^{2-\iota}\right)\right]. \tag{10}
$$

Now, we can apply one more set of set of transformations $\tilde{X} = \frac{2}{\underline{d}}\sqrt{t}$, $\tilde{x} = \frac{2}{\underline{d}}\sqrt{\tau}$, $\tilde{Y} = u + bt$, and $\tilde{y} = \mu + b\tau$, where $b = 3\underline{d} - \frac{\underline{d}^2}{2} + \frac{\underline{d}^3}{6}$ is chosen to eliminate the $\tilde{X}^2$ term from the $\tilde{y}$ equation.

$$
\begin{aligned}
\tilde{x} &= \tilde{X} - \tilde{X}^2 - \frac{1}{\underline{d}^3}\tilde{X}\tilde{Y} + O(|(\tilde{X},\tilde{Y})|^3) \\
\tilde{y} &= -\tilde{Y} - (2+\underline{d})\tilde{X}\tilde{Y} + O(|(\tilde{X},\tilde{Y})|^3)
\end{aligned} \tag{11}
$$

This mapping and its inverse:

$$\begin{aligned} \tilde{X} &= \tilde{x}+\tilde{x}^2-\frac{1}{\underline{d}^3}\tilde{x}\tilde{y}+O(|(\tilde{x},\tilde{y})|^3) \\ \tilde{Y} &= -\tilde{y}+(2+\underline{d})\tilde{x}\tilde{y}+O(|(\tilde{x},\tilde{y})|^3) \end{aligned} \tag{12}$$

are in form for direct application of the results in Fefferman (2021).

Applying the above series of transformations directly to the map $T:(F,y)\mapsto(\nu,\eta)$ in (8) yields the mapping $\Psi:(\tilde{X},\tilde{Y})\mapsto(\tilde{x},\tilde{y})$ given by

$$\tilde{x} = \frac{2}{\underline{d}}\sqrt{\nu\left(\frac{\underline{d}^2}{4}\tilde{X}+\underline{d},\frac{4}{\underline{d}^2\tilde{X}^2}(\tilde{Y}+\underline{d}^3)-b\right)-\underline{d}} \tag{13}$$

$$\tilde{y} = \left[\nu\left(\frac{\underline{d}^2}{4}\tilde{X}+\underline{d},\frac{4}{\underline{d}^2\tilde{X}^2}(\tilde{Y}+\underline{d}^3)-b\right)-\underline{d}\right]\left[\eta\left(\frac{\underline{d}^2}{4}\tilde{X}+\underline{d},\frac{4}{\underline{d}^2\tilde{X}^2}(\tilde{Y}+\underline{d}^3)-b\right)+b\right]-\underline{d}^3$$

So Ψ is the mapping approximated by (11) and the inverse $\Psi^{-1}:(X,Y)\mapsto(x,y)$ is approximated in (12).

By Fefferman (2021) Theorem 1.1, there exists a function $\bar{c}$ that:
(a) generates an invariant curve for Ψ, for $\bar{\Gamma}=\{(\tilde{x},\bar{c}(\tilde{x}))\,|\,\tilde{x}\in[0,\bar{\delta}]\}$, $\Psi(\bar{\Gamma})\subset\bar{\Gamma}$;
(b) is tangent to the x-axis near the fixed point at the origin, $\bar{c}(\tilde{x})=O(\tilde{x}^3)$ as $\tilde{x}\downarrow 0$; and
(c) is infinitely differentiable on $[0,\bar{\delta}]$ for some $\bar{\delta}>0$.
This theorem also delivers uniqueness in the following sense. Let $\breve{c}$ be a function such that $\tilde{x}^{-\frac{2}{3}}\breve{c}(\tilde{x})\to 0$ as $\tilde{x}\downarrow 0$ and define $\breve{\Gamma}=\{(\tilde{x},\breve{c}(\tilde{x}))\,|\,\tilde{x}\in[0,\breve{\delta}]\}$ for $\breve{\delta}>0$. If $\Psi(\breve{\Gamma})\subset\breve{\Gamma}$, then $\breve{c}=\bar{c}$ on $[0,\tilde{\delta}]$ for some $\tilde{\delta}>0$.

Given the function $\bar{c}$ that defines an invariant curve for Ψ, we define a corresponding function for T:

$$c_\alpha(F)=\frac{\bar{c}\left(\frac{2}{\underline{d}}\sqrt{F-\underline{d}}\right)+\underline{d}^3}{F-\underline{d}}-b.$$

for $F\neq\underline{d}$ such that $\bar{c}\left(\frac{2}{\underline{d}}\sqrt{F-\underline{d}}\right)$ is well defined. Then, c_α will inherit the smoothness properties of $\bar{c}$ on this domain proving (iii). Consider F such that $\underline{d}<F\leq\underline{d}+\frac{\underline{d}^2}{4}\bar{\delta}^2$, and define $y=c_\alpha(F)$. Now apply the map T yielding (ν,η) as given by (8). To show that c_α defines an invariant curve for T, we need to show that $\eta=c_\alpha(\nu)$. Let $\tilde{X}=\frac{2}{\underline{d}}\sqrt{F-\underline{d}}$ and $\tilde{Y}=(y+b)(F-\underline{d})-\underline{d}^3$. By the definition of c_α, $\tilde{Y}=\bar{c}(\tilde{X})$ and $\tilde{X}\in(0,\bar{\delta}]$. Define $(\tilde{x},\tilde{y})=\Psi(\tilde{X},\tilde{Y})$ as in (8). Then, the result in Fefferman

(2021) shows that $\tilde{y} = \bar{c}(\tilde{x})$ and $\tilde{x} \in (0, \bar{\delta}]$. Notice that $\nu = \frac{d^2}{4}\tilde{x} + \underline{d} \in (\underline{d}, \underline{d} + \frac{d^2}{4}\bar{\delta}^2]$ and

$$\eta = \frac{4}{\underline{d}^2\tilde{x}^2}(\tilde{y} + \underline{d}^3) - b = \frac{4}{\underline{d}^2\tilde{x}^2}(\bar{c}(\tilde{x}) + \underline{d}^3) - b = \frac{\bar{c}\left(\frac{2}{\underline{d}}\sqrt{\nu - \underline{d}}\right) + \underline{d}^3}{\nu - \underline{d}} - b = c_\alpha(\nu),$$

as desired. This invariance shows that c_α satisfies (7) for $F \in (\underline{d}, \underline{d} + \frac{d^2}{4}\bar{\delta}^2]$. Also, note that by the definition of c_α, $\bar{c}(\tilde{x}) = O(\tilde{x}^3)$ directly implies $c_\alpha(F) - \left(\frac{d^3}{F - \underline{d}} - b\right) = O(\sqrt{F - \underline{d}})$ as $F \downarrow \underline{d}$.

Next, we show uniqueness of c_α. Consider a function

$$\breve{c}_\alpha \in \left\{ c \,\middle|\, (F - \underline{d})^{1/3}\left[c(F) - \left(\frac{\underline{d}^3}{F - \underline{d}} - b\right)\right] \to 0 \text{ as } F \downarrow \underline{d} \right\}$$

such that $T(\Pi) \subset \Pi$ where $\Pi = \{(F, \breve{c}_\alpha(F)) \,|\, F \in (\underline{d}, \underline{d} + \frac{d^2}{4}\breve{\delta}^2]\}$ for some $\breve{\delta} > 0$. Set $\breve{c}(\tilde{x}) = \left[\breve{c}_\alpha\left(\frac{d^2}{4}\tilde{x}^2 + \underline{d}\right) + b\right]\left(\frac{d^2}{4}\tilde{x}^2\right) - \underline{d}^3$. Similar to the argument above, $T(\Pi) \subset \Pi$ implies that $\Psi(\breve{\Gamma}) \subset \breve{\Gamma}$ for $\breve{\Gamma} = \{(\tilde{x}, \breve{c}(\tilde{x})) \,|\, \tilde{x} \in (0, \breve{\delta}]\}$. By the uniqueness result in Fefferman (2021), it follows that $\breve{c} = \bar{c}$ on $(0, \tilde{\delta}]$ for some $\tilde{\delta} > 0$ and hence $\breve{c}_\alpha = c_\alpha$ on $(\underline{d}, \underline{d} + \frac{d^2}{4}\tilde{\delta}^2]$.

Now, we show that $c_\alpha(F)$ is decreasing for $F \in (\underline{d}, \underline{d} + \delta_3]$ for some $\delta_3 > 0$. Since $\bar{c}'(\tilde{x})$ is continuous on $[0, \bar{\delta}]$, it is also bounded. In particular, $\bar{c}'(\tilde{x}) < k$ on $[0, \bar{\delta}]$, for some $k > 0$. Since $\bar{c}(\tilde{x}) = O(\tilde{x}^3)$, there exists $\bar{\delta}_1 \in (0, \bar{\delta}]$ such that $\bar{c}(\tilde{x}) > -\frac{d^3}{3}$ for $\tilde{x} \in [0, \bar{\delta}_1]$, and hence $-\underline{d}\,\bar{c}\left(\frac{2}{\underline{d}}\sqrt{F - \underline{d}}\right) < \frac{d^4}{3}$ for $F \in (\underline{d}, \underline{d} + \frac{d^2}{4}\bar{\delta}_1^2]$. Let $\delta_3 = \min\{\frac{d^2}{4}\bar{\delta}_1, \frac{d^8}{9k^2}\}$. Then, $F \in (\underline{d}, \underline{d} + \delta_3]$ implies $\sqrt{F - \underline{d}} < \frac{d^4}{3k}$ and $\bar{c}'\left(\frac{2}{\underline{d}}\sqrt{F - \underline{d}}\right)\sqrt{F - \underline{d}} < \frac{d^4}{3}$. Then, for $F \in (\underline{d}, \underline{d} + \delta_3]$,

$$c_\alpha'(F) = \frac{\bar{c}'\left(\frac{2}{\underline{d}}\sqrt{F - \underline{d}}\right)\sqrt{F - \underline{d}} - \underline{d}\bar{c}\left(\frac{2}{\underline{d}}\sqrt{F - \underline{d}}\right) - \underline{d}^4}{\underline{d}(F - \underline{d})^2} < \frac{\frac{d^4}{3} + \frac{d^4}{3} - \underline{d}^4}{\underline{d}(F - \underline{d})^2} = -\frac{\underline{d}^3}{3(F - \underline{d})^2} < 0.$$

Lastly, take any $k > 0$. Set $\delta_2 = \min\left\{\frac{2\underline{d}^3}{3(k+b)}, \frac{d^2}{4}\bar{\delta}_1^2\right\}$. For $F \in (\underline{d}, \underline{d} + \delta_2]$, $0 < \frac{2}{d}\sqrt{F - \underline{d}} \le \bar{\delta}_1$ which implies $\bar{c}\left(\frac{2}{\underline{d}}\sqrt{F - \underline{d}}\right) > -\frac{\underline{d}^3}{3}$, and $F - \underline{d} < \frac{2\underline{d}^3}{3(k+b)}$ implies $-d^3 + (k+b)(F - \underline{d}) < -\frac{d^3}{3}$. Hence, $\bar{c}\left(\frac{2}{\underline{d}}\sqrt{F - \underline{d}}\right) > -d^3 + (k+b)(F - \underline{d})$ which can be re-arranged to yield $c_\alpha(F) > k$, so (iv) is proven. ⊠

The function $c_\alpha(F)$ given in Lemma 9 is well specified for a neighborhood $F \in (\underline{d}, \underline{d}+\delta]$. To be useful as a critical value function for controlling the size of the test statistic t^2, c_α needs to be extended to $F > \underline{d}+\delta$. In practice we start this extension by applying the map T^{-1} (defined in the proof of Lemma 9). Suppose $(F_1, y_1) = T^{-1}(\underline{d}+\delta, c_\alpha(\underline{d}+\delta))$. We can define $c_\alpha(F)$ on $(\underline{d}+\delta, F_1]$ so that $T^{-1}(\{(F, c_\alpha(F)) \,|\, F \in (\underline{d}, \underline{d}+\delta]\}) = \{(F, c_\alpha(F)) \,|\, F \in (\underline{d}, F_1]\}$. One could continue to iterate this map and extend the function c_α. However, at some point the mapping T will no longer characterize size of the critical value function under $\rho = 1$ (see Property 1(c) in the definition of tF). Our approach will be to stop iterating and simply flatten the critical value function before encountering this problem. Potentially other approaches could be taken, so we make the following assumption that accommodates a wide range of possibilities.

Assumption 4. Suppose $c_\alpha(F)$ is well-defined for $F > \underline{d}$. Assume that for some $\tilde{\delta} > 0$, c_α is decreasing on $(\underline{d}, \underline{d}+\tilde{\delta}]$ and $c_\alpha(F) \leq c_\alpha(\underline{d}+\tilde{\delta})$ for $F > \underline{d}+\tilde{\delta}$.

From Lemma 9(iv), we know that c_α, as given by that result, is decreasing for some segment local to $\underline{d}$, so the key property supposed in Assumption 4 is a ceiling on the extended critical value function. That is, the behavior of c_α as it asymptotes at $\underline{d}$ is provided formally by Lemma 9, and it is straightforward to directly check Assumption 4 for the whatever approach is taken to extending c_α. For our extension approach, we verify the ceiling in Assumption 4 numerically and find it to be trivially satisfied.

Lemma 10. *Suppose Assumption 4 holds, and suppose $c_\alpha(\cdot)$ satisfies (7) for $F \in (\underline{d}, \underline{d}+\delta]$ for some $\delta > 0$ with properties given in Lemma 9. Then, for some $\delta_0 > 0$,*

$$\Pr_{f_0, \rho=1}(t^2 \geq c_\alpha(f^2)) = \alpha \quad \textit{for } 0 < f_0 \leq \delta_0 \tag{14}$$

PROOF: Given $c_\alpha(\cdot)$ satisfying (7) for $F \in (\underline{d}, \underline{d}+\delta]$, we can set

$$f_0 = \frac{F}{\sqrt{c_\alpha(F)} + \sqrt{F}},$$

and

$$\begin{aligned} f^L &= \Phi^{-1}\left(\Phi\left(\sqrt{F} - f_0\right) - (1-\alpha)\right) + f_0 \\ &= \Phi^{-1}\left(\Phi\left(\sqrt{F} - \frac{F}{\sqrt{c_\alpha(F)} + \sqrt{F}}\right) - (1-\alpha)\right) + \frac{F}{\sqrt{c_\alpha(F)} + \sqrt{F}}. \end{aligned} \tag{15}$$

The definitions of f_0 and f^L and equation (7) ensure that

$$\begin{aligned} c_\alpha(F) &= \frac{F(\sqrt{F}-f_0)^2}{f_0^2} \\ (1-\alpha) &= \Phi(\sqrt{F}-f_0) - \Phi(f^L - f_0) \\ c_\alpha((f^L)^2) &= \frac{(f^L)^2(f^L-f_0)^2}{f_0^2}. \end{aligned}$$

When $\rho = 1$,

$$t^2(f) = \frac{f^2(f-f_0)^2}{f_0^2}.$$

and we can see that $f = \sqrt{F}$ and f^L are intersection points of $t^2(f)$ and $c_\alpha(f^2)$. Next we will show that these points of intersection fully characterize the set $\{f \,|\, t^2(f) \geq c_\alpha(f^2)\}$ for small values of f_0.

For $F \in (\underline{d}, \underline{d}+\tilde{\delta}]$, $f_0 = \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}$ is monotonically increasing in F and $c_\alpha(F) \to \infty$ as $F \downarrow \underline{d}$ since by Lemma 9(iv). So $\underline{d} < F \leq \underline{d}+\tilde{\delta}$ implies $0 < f_0 \leq \tilde{\delta}_0$ where $\tilde{\delta}_0 = \frac{\underline{d}+\tilde{\delta}}{\sqrt{c_\alpha(\underline{d}+\tilde{\delta})}+\sqrt{\underline{d}+\tilde{\delta}}}$. Take a value $f_0 \in (0, \tilde{\delta}_0]$. There is exactly one value of $F \in (\underline{d}, \underline{d}+\tilde{\delta}]$ such that $f_0 = \frac{F}{\sqrt{c_\alpha(F)}+\sqrt{F}}$. Choose $\tilde{\delta}_1 \in (0, \tilde{\delta}_0]$ such that for each $f_0 \in (0, \tilde{\delta}_1]$, there is a well-defined f^L from (15) such that $(f^L)^2 \in (\underline{d}, \underline{d}+\tilde{\delta}]$.

Let $\tilde{\delta}_2 = \min\{\tilde{\delta}_1, \sqrt{\underline{d}}\}$. Since $c_\alpha(f^2)$ is undefined (or infinite) for $f^2 \leq \underline{d}$, $c_\alpha(f^2)$ cannot intersect $t^2(f)$ for $0 \leq f \leq f_0$ if $f_0 \in (0, \tilde{\delta}_2]$. From the definitions, we know that $F > f_0^2$ and $f^L < f_0$. For any $f_0 \in (0, \tilde{\delta}_2]$, $c_\alpha(f^2)$ and $t^2(f)$ do not intersect on $f \in (0, f_0)$ and so we must also have $f^L \leq 0$.

Again consider any value $f_0 \in (0, \tilde{\delta}_2]$ with corresponding F and f^L. For $f > f_0$, $t^2(f)$ is strictly increasing and $c_\alpha(f^2)$ is decreasing for $f \in (f_0, F]$. For $f > F$, $t^2(f) > t^2(\sqrt{F}) = c_\alpha(F) \geq c_\alpha(f^2)$. Similarly, $t^2(f) < c_\alpha(f^2)$ for $f^L < f < 0$ and $t^2(f) > c_\alpha(f^2)$ for $f < f^L$. It follows that $\{f \,|\, t^2(f) \geq c_\alpha(f^2)\} = \{f \leq f^L\} \cup \{f \geq \sqrt{F}\}$ and

$$\Pr_{f_0,\rho=1}(t^2 \geq c_\alpha(f^2)) = 1 - \Big(\Phi(\sqrt{F}-f_0) - \Phi(f^L - f_0)\Big) = \alpha$$

$\boxtimes$

B.2 Non-existence of "smaller" critical value function

Corollary (non-existence of "smaller" critical value function): Consider any alternative function $k(F)$ satisfying properties 1(a) and 1(c), and such that $k(F) \leq c_\alpha(F)$ for all F , with $k(F) < c_\alpha(F)$ for some value of F. Then $k(F)$ cannot control size to be α.

PROOF: Since both c_α and k satisfy Property 1(a), there are $\tilde{F}_c$ and $\tilde{F}_k$ such that $c_\alpha(F) = \tilde{c}(F, \tilde{F}_c)$ and $k(F) = \tilde{c}(F, \tilde{F}_k)$. By supposition there is some point F_1 such that $k(F_1) < c_\alpha(F_1)$.

Suppose there does not exist $F_1 \leq \tilde{F}_c$ with $k(F_1) < c_\alpha(F_1)$. Then, we must have $k(F) = c_\alpha(F)$ for all $F \leq \tilde{F}_c$ and there must be an $F_1 > \tilde{F}_c$ with $k(F_1) < c_\alpha(F_1)$. In this case, $\tilde{F}_c < \tilde{F}_k$ and $\tilde{c}(F, \tilde{F}_c) > \tilde{c}(F, \tilde{F}_k)$, otherwise c_α and k would be identical functions. But this would contradict $\tilde{F}_c$ being determined by the maximization in Property 2. We conclude that there does exist $F_1 \leq \tilde{F}_c$ with $k(F_1) < c_\alpha(F_1)$.

Define

$$f_0^* = \frac{F_1}{\sqrt{c_\alpha(F_1)} + \sqrt{F_1}}.$$

And recall that when $\rho = 1$, $t^2(F) = \frac{F\left(\sqrt{F} - f_0\right)^2}{f_0^2}$. Fixing $f_0 = f_0^*$, $t^2(F_1) = c_\alpha(F_1) > k(F_1)$, where c_α and k are continuous and non-increasing at F_1, and t^2 is continuous and strictly increasing at F_1 (since $F_1 > f_0^*$). It follows that there exists an ε such that $0 < \varepsilon < F_1 - \underline{d}$ and $k(F) < t^2(F) < c_\alpha(F)$ for $F \in [F_1 - \varepsilon, F_1)$. Then, $[F_1 - \varepsilon, F_1) \subset \{F \,|\, t^2(F) \geq k(F)\}$ while $[F_1 - \varepsilon, F_1) \cap \{F \,|\, t^2(F) \geq c_\alpha(F)\} = \emptyset$. Since $k(F) \leq c_\alpha(F)$ for all F, $\{F \,|\, t^2(F) \geq c_\alpha(F)\} \cup [F_1 - \varepsilon, F_1) \subset \{F \,|\, t^2(F) \geq k(F)\}$. Finally,

$$\begin{aligned}
\alpha &= \Pr_{f_0^*, \rho=1}(t^2(F) \geq c_\alpha(F)) \\
&\leq \Pr_{f_0^*, \rho=1}(t^2(F) \geq k(F)) - \Pr_{f_0^*, \rho=1}(F \in [F_1 - \varepsilon, F_1)) \\
&< \Pr_{f_0^*, \rho=1}(t^2(F) \geq k(F))
\end{aligned}$$

But this contradicts k controlling size at level α. We conclude that there does not exist a "smaller" critical value function than c_α. ☒

B.3 tF: Size control for $|\rho|$ near 1, small f_0

Result: Under the null hypothesis, for any arbitrarily small departure from $|\rho| = 1$ there exists a neighborhood of values f_0 near $f_0 = 0$ such that all rejection probabilities $\Pr_{\rho, f_0}\left[t^2 > c_\alpha(F)\right]$ are smaller than the intended significance level α.

PROOF: Below, the proof involves focusing on small f_0, using the change of variables $\varrho = \sqrt{1-\rho^2}$ and considering the derivative of the rejection probability with respect to ϱ, evaluated at $\varrho = 0$. We find that the first derivative is zero for f_0 small. We therefore compute the second derivative at $\varrho = 0$, and then take a Taylor series expansion of this second derviative expression to find that in a neighborhood of $f_0 = 0$, this second derivative is negative, which implies that when one departs slightly from $|\rho| = 1$, then the rejection probability will decline, leading to size control in this "corner" of the nuisance parameter space. Below, we suppress α to simplify notation.

We begin with our relationship

$$t^2 = \frac{f^2 t_{AR}^2}{f^2 - 2\rho t_{AR} f + t_{AR}^2}$$

which expresses t^2 as a function of t_{AR}, f, and correlation ρ.

Under the tF procedure, rejection occurs in the event that

$$f^2 t_{AR}^2 - \left(f^2 - 2\rho t_{AR} f + t_{AR}^2\right) c\left(f^2\right) > 0$$

where $c\left(f^2\right)$ is our critical value function, and where f and t_{AR} are bivariate normal with unit variances and mean vector $(f_0, 0)$ (under the null hypothesis), with correlation ρ.

We do a change of variables

$$x = f - \rho t_{AR}$$

and note that x and t_{AR} are by construction uncorrelated and therefore, by bivariate normality, independent. x has mean f_0 and variance $1-\rho^2$.

Substituting, we now have rejection occurring when

$$(x+\rho t_{AR})^2 t_{AR}^2 - \left((x+\rho t_{AR})^2 - 2\rho t_{AR}(x+\rho t_{AR}) + t_{AR}^2\right) c\left((x+\rho t_{AR})^2\right) > 0 \quad (16)$$

We now have

$$\begin{aligned}\Pr\left[t^2 > c\left(f^2\right)\right] = \int_{-\infty}^{\infty} & [1 - \Phi(r_4(\rho, z)) \\ & + \Phi(r_1(\rho, z)) \\ & + 1[|z| > \bar{z}]\{\Phi(r_3(\rho, z)) - \Phi(r_2(\rho, z))\}] \frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{z - f_0}{\sqrt{1-\rho^2}}\right) dz\end{aligned}$$

where r_1, r_2, r_3, r_4 are functions of x and ρ that are implicitly defined by the r_j that satisfy

$$(x+\rho r_j)^2 r_j^2 - \left((x+\rho r_j)^2 - 2\rho r_j(x+\rho r_j) + r_j^2\right) c\left((x+\rho r_j)^2\right) = 0$$

r_j gives the t_{AR} coordinate of any point on the critical value boundaries, as a function of ρ and x. Since the equation defines a (near) quartic polynomial in r_j, we can expect up to four roots of the equation. z is the variable of integration for the random variable x.

We now do two changes of variables

$$U = \frac{x - f_0}{\sqrt{1-\rho^2}}$$
$$\varrho = \sqrt{1-\rho^2}$$

where we will be focusing on a neighborhood, without loss of generality, of $\rho = 1$ (and equivalently a neighborhood of $\varrho = 0$).

t_{AR} and U are also independent; U is a standard normal random variable. With this change of variables we substitute and now have

$$\begin{aligned}\Pr\left[t^2 > c\left(f^2\right)\right] = \int_{-\infty}^{\infty} & [1 - \Phi(r_4^*(\varrho, u, f_0)) \\ & + \Phi(r_1^*(\varrho, u, f_0)) \\ & + 1\left[|f_0 + \varrho u| > \bar{z}\right]\{\Phi(r_3^*(\varrho, u, f_0)) - \Phi(r_2^*(\varrho, u, f_0))\}]\,\phi(u)\,du\end{aligned}$$

where we have $r_j^*(\varrho, u, f_0) = r_j\left(\sqrt{1-\varrho^2}, f_0 + \varrho u\right)$ for $j = 1,2,3,4$, and $\bar{z}$ is defined as the value of u that separates the regions where there are 4 or 2 roots. Note that, using the change of variables, each of the r_j^* also satisfy the equation

$$\begin{aligned}F\left(\varrho, r_j^*, u, f_0\right) = &\left(f_0 + \varrho u + \sqrt{1-\varrho^2} r_j^*\right)^2 \left(r_j^*\right)^2 \\ - &\left(\left(f_0 + \varrho u + \sqrt{1-\varrho^2} r_j^*\right)^2 - 2\sqrt{1-\varrho^2} r_j^*\left(f_0 + \varrho u + \sqrt{1-\varrho^2} r_j^*\right) + \left(r_j^*\right)^2\right) \\ & c\left(\left(f_0 + \varrho u + \sqrt{1-\varrho^2} r_j^*\right)^2\right) = 0\end{aligned}$$

Derivatives: first and second derivatives

We now take both the first and second derivative of the rejection probability with respect to ϱ, evaluated at $\varrho = 0$, and with f_0 "sufficiently small". Here, "sufficiently small" corresponds to small enough f_0 so that the derivative terms below associated with r_2^* and r_3^* will be zero.

Thus, with sufficiently small f_0, the first derivative of the rejection probability is

$$\begin{aligned}\frac{\partial \Pr\left[t^2 > c\left(f^2\right)\right]}{\partial\ \varrho} = \int_{-\infty}^{\infty} \Bigg[& -\phi\left(r_4^*\right)\frac{\partial r_4^*}{\partial\ \varrho} \\ & + \phi\left(r_1^*\right)\frac{\partial r_1^*}{\partial\ \varrho}\Bigg]\phi\left(u\right)du\end{aligned}$$

and the second derivative is

$$\begin{aligned}\frac{\partial^2 \Pr\left[t^2 > c\left(f^2\right)\right]}{\partial\ \varrho^2} &= \int_{-\infty}^{\infty}\Bigg[r_4^*\phi\left(r_4^*\right)\left(\frac{\partial r_4^*}{\partial\ \varrho}\right)^2 - \phi\left(r_4^*\right)\frac{\partial^2 r_4^*}{\partial\ \varrho^2} \\ & -r_1^*\phi\left(r_1^*\right)\left(\frac{\partial r_1^*}{\partial\ \varrho}\right)^2 + \phi\left(r_1^*\right)\frac{\partial^2 r_1^*}{\partial\ \varrho^2}\Bigg]\phi\left(u\right)du \\ &= \int_{-\infty}^{\infty}\Bigg[\phi\left(r_4^*\right)\left\{r_4^*\left(\frac{\partial r_4^*}{\partial\ \varrho}\right)^2 - \frac{\partial^2 r_4^*}{\partial\ \varrho^2}\right\} \\ & -\phi\left(r_1^*\right)\left\{r_1^*\left(\frac{\partial r_1^*}{\partial\ \varrho}\right)^2 - \frac{\partial^2 r_1^*}{\partial\ \varrho^2}\right\}\Bigg]\phi\left(u\right)du\end{aligned}$$

We then take the following steps:

1. Using implicit differentiation, obtain the first and second derivatives of r_j^* with respect to ϱ. These expressions will be functions of $r_j^*, \varrho, u, f_0, c\left(\cdot\right)$, and $c'\left(\cdot\right)$.

2. Evaluate these derivatives at $\varrho = 0$. The expressions will be functions of $r_j^*, u, f_0, c\left(\cdot\right)$, and $c'\left(\cdot\right)$

3. Because $\varrho = 0$ is equivalent to $\rho = 1$, we can replace $r_j^* = f_j^* - f_0$, where f_j^* is the corresponding f-coordinate on the critical value boundary. This substitution results in functions that involve $f_j^*, u, f_0, c\left(\cdot\right)$, and $c'\left(\cdot\right)$

4. We use the fact that at $\varrho = 0$, that for every associated f_0 there are f_j^* that satisfy $f_0 = \frac{\left(f_j^*\right)^2}{\sqrt{\left(f_j^*\right)^2 + \sqrt{c\left(\left(f_j^*\right)^2\right)}}}$, substituting this in leaves expressions that involve $f_j^*, u, c\left(\cdot\right)$, and $c'\left(\cdot\right)$

5. We make another substitution: $\zeta = c\left(\left(f_j^*\right)^2\right)\left[\left(f_j^*\right)^2 - q\right]$ which implies that $c'\left((f^*)^2\right) = \frac{\zeta'\left((f^*)^2\right)}{(f^*)^2 - q} - \frac{\zeta\left((f^*)^2\right)}{\left((f^*)^2 - q\right)^2}$. This substituion leads to expressions that are functions of $f_j^*, u, \zeta(\cdot)$, and $\zeta'(\cdot)$

6. Another change of variables, using $\tau > 0$ as $f_4^* = \sqrt{\tau^2 + q}$ and $f_1^* = -\sqrt{\tau^2 + q}$, leaving expressions that involve τ, u, ζ, ζ', and q.

7. Collect powers of u, integrate out u, noting U is standard normal, so that $\int u^2 \phi(u)\, du = 1$. This leaves expressions that involve τ, ζ, and ζ', and q. At this step, we have found that the first derivative of the rejection probability for f_0 sufficiently small as described above is equal to zero.

8. Take a first order taylor series expansion in τ around $\tau = 0$, and note from property (i) from Lemma 9 that as τ tends to zero, ζ tends to q^3 and ζ' tends to $-\left(3q - \frac{q^2}{2} + \frac{q^3}{6}\right)$. This means that the linear approximation for the second derivative (with respect to ϱ) is a linear function with constants and linear coefficient depending on q only.

9. Specifically, the second derivative is

$$\phi(\sqrt{q})\left[-2\left(\sqrt{q} + q^{\frac{3}{2}}\right) - \frac{4(1+q)}{\sqrt{q}}\tau\right] \tag{17}$$

This means that we can always find a small enough $\tau = \sqrt{F - q}$ so that the second derivative is negative. Since we are at $\varrho = 0$, for each of these small values of τ, there is a corresponding f_0

$$f_0 = \frac{\tau^2 + q}{\sqrt{c(\tau^2 + q)} + \sqrt{\tau^2 + q}}$$

(f_0 and τ are one-to-one with sufficiently small τ, because

$$\frac{df_0}{d\tau} = \frac{2\tau\left(\sqrt{c(\tau^2 + q)} + \sqrt{\tau^2 + q}\right) - \left(\tau^2 + q\right)\left[\frac{c'}{2\sqrt{c}}2t - \frac{2\tau}{2\sqrt{\tau^2 + q}}\right]}{\left(\sqrt{c(\tau^2 + q)} + \sqrt{\tau^2 + q}\right)^2} > 0$$

for all small postive values of τ, since c' is negative).

So this means that you can always find a neighborhood $(0, \tau_0)$ such that for all values of τ in the neighborhood, the second derivative will be negative, and therefore, you can always find a neighborhood $(0, f_0^*)$ such that for all f_0 in the neighborhood, the second derivative will be negative. We have cross-checked the expression in (17) by numerically computing rejection probabilities for ρ values close to 1 and $f_0 = 0$. ⊠

C Conditional Expected Length: *AR* and *tF*

C.1 Limiting Distribution of *AR* and *tF* confidence sets

Derivation of inflation factor $\frac{\sqrt{1-\frac{q}{\hat{F}}\left(1-\hat{\hat{\rho}}^2\right)}}{1-\frac{q}{\hat{F}}}$

To derive how much we inflate the 2SLS confidence interval to obtain the *AR* interval length, we use the relationship

$$\hat{t}_{AR}^2 = \frac{\hat{t}^2 \hat{f}^2}{\hat{f}^2 + 2\hat{\hat{\rho}}\sqrt{\hat{F}}\hat{t} + \hat{t}^2}$$

and solve

$$\frac{\hat{t}^2 \hat{f}^2}{\hat{f}^2 + 2\hat{\hat{\rho}}\sqrt{\hat{F}}\hat{t} + \hat{t}^2} < q$$

for $\hat{t}$.

$$\hat{t}^2 \hat{f}^2 - q\left(\hat{f}^2 + 2\hat{\hat{\rho}}\sqrt{\hat{F}}\hat{t} + \hat{t}^2\right) < 0$$
$$\hat{t}^2 \hat{f}^2 - q\hat{f}^2 - q2\hat{\hat{\rho}}\sqrt{\hat{F}}\hat{t} - q\hat{t}^2 < 0$$
$$\left(\hat{f}^2 - q\right)\hat{t}^2 - \left(2q\hat{\hat{\rho}}\sqrt{\hat{F}}\right)\hat{t} - q\hat{f}^2 < 0$$

which is a convex function in $\hat{t}$ when $\hat{f}^2 > q$. So the $\hat{t}$ that satisfies the inequality is an interval in this case, with endpoints

$$
\begin{aligned}
\frac{\left(2q\hat{\rho}\sqrt{\hat{F}}\right) \pm \sqrt{\left(2q\hat{\rho}\sqrt{\hat{F}}\right)^2 + 4\left(\hat{f}^2 - q\right) q\hat{f}^2}}{2\left(\hat{f}^2 - q\right)} &= \\
\frac{\left(q\hat{\rho}\sqrt{\hat{F}}\right) \pm \sqrt{q}\sqrt{\hat{F}}\sqrt{q\hat{\rho}^2 + \hat{f}^2 - q}}{\left(\hat{f}^2 - q\right)} &= \\
\frac{\left(q\hat{\rho}\sqrt{\hat{F}}\right) \pm \sqrt{q}\sqrt{\hat{F}}\sqrt{\hat{f}^2 - q\left(1 - \hat{\rho}^2\right)}}{\left(\hat{f}^2 - q\right)} &= \\
\frac{\left(\frac{q\hat{\rho}}{\sqrt{\hat{F}}}\right) \pm \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\rho}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)} &
\end{aligned}
$$

Since

$$\hat{t} = \frac{\hat{\beta} - \beta}{\sqrt{\hat{V}_N\left(\hat{\beta}\right)}}$$

then the *AR* interval is given by

$$\hat{\beta} + \frac{-\left(\frac{q\hat{\rho}}{\sqrt{\hat{F}}}\right) + \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\rho}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{\hat{V}_N\left(\hat{\beta}\right)} \geq \beta \geq \hat{\beta} + \frac{-\left(\frac{q\hat{\rho}}{\sqrt{\hat{F}}}\right) - \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\rho}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{\hat{V}_N\left(\hat{\beta}\right)}$$

$$\hat{\beta} + \frac{-\sqrt{\frac{q}{\hat{F}}}\hat{\rho} - \sqrt{1 - \frac{q\left(1-\hat{\rho}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{q}\sqrt{\hat{V}_N\left(\hat{\beta}\right)} \leq \beta \leq \hat{\beta} + \frac{-\sqrt{\frac{q}{\hat{F}}}\hat{\rho} + \sqrt{q}\sqrt{1 - \frac{q\left(1-\hat{\rho}^2\right)}{\hat{F}}}}{\left(1 - \frac{q}{\hat{F}}\right)}\sqrt{q}\sqrt{\hat{V}_N\left(\hat{\beta}\right)}$$

Since the half-length of the 2SLS confidence interval is $\sqrt{q}\sqrt{\hat{V}_N\left(\hat{\beta}\right)}$, then the inflation factor to obtain the half-length of the *AR* interval is

$$\frac{\sqrt{1 - \frac{q}{\hat{F}}\left(1 - \hat{\rho}^2\right)}}{\left(1 - \frac{q}{\hat{F}}\right)}$$

Derivation of limiting distributions of the $(1-\alpha)$ confidence intervals $\hat{L}_{IV}, \hat{L}_{AR}, \hat{L}_{tF}$

$$\hat{L}_{IV} \xrightarrow{d} L_{IV} \equiv 2\sqrt{q_{1-\alpha}}\sqrt{1-2\rho\frac{t_{AR}(\beta)}{f}+\frac{t_{AR}^2(\beta)}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_\Omega} \tag{18}$$

$$\hat{L}_{AR} \xrightarrow{d} L_{AR} \equiv \frac{\sqrt{F}\sqrt{F-q_{1-\alpha}(1-\tilde{\rho}^2)}}{F-q_{1-\alpha}}L_{IV}$$

$$\hat{L}_{tF} \xrightarrow{d} L_{tF} \equiv \frac{\sqrt{c_\alpha(F)}}{\sqrt{q_{1-\alpha}}}L_{IV}$$

where

$$\tilde{\rho}^2 = \frac{(-t_{AR}(\beta)+\rho f)^2}{\left(f^2-2\rho t_{AR}(\beta)f+t_{AR}^2(\beta)\right)}$$

$$V_\Omega = \frac{AV\left(\widehat{\pi\beta}\right)-2\beta AC\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2 AV(\hat{\pi})}{AV(\hat{\pi})}$$

Limiting Distribution of $\hat{L}_{IV}$

Throughout this proof, when we consider the statistics $\hat{t}$ and $\hat{t}_{AR}$, they have $\left(\hat{\beta}-\beta\right)$, the estimator minus the true value of the parameter β, in the numerator.

By definition we have

$$\hat{L}_{IV} = 2\sqrt{q}\sqrt{\frac{\hat{V}_N\left(\widehat{\pi\beta}\right)-2\hat{\beta}C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\hat{\beta}^2\hat{V}_N(\hat{\pi})}{\hat{\pi}^2}}$$

We first note that

$$\frac{\hat{V}_N\left(\widehat{\pi\beta}\right)-2\hat{\beta}C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\hat{\beta}^2\hat{V}_N(\hat{\pi})}{\hat{\pi}^2}=\frac{\left(\hat{\beta}-\beta\right)^2}{\hat{t}^2}$$

$$=\frac{1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}}{\hat{t}_{AR}^2}\left(\hat{\beta}-\beta\right)^2$$

$$=\frac{1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}}{\hat{t}_{AR}^2}\frac{\hat{t}_{AR}^2}{\hat{\pi}^2}\left(\hat{V}_N\left(\widehat{\pi\beta}\right)-2\beta C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2\hat{V}_N(\hat{\pi})\right)$$

$$=\left(1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}\right)\frac{\hat{V}_N(\hat{\pi})}{\hat{V}_N(\hat{\pi})}\frac{\left(\hat{V}_N\left(\widehat{\pi\beta}\right)-2\beta C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2\hat{V}_N(\hat{\pi})\right)}{\hat{\pi}^2}$$

$$=\left(1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}\right)\frac{1}{\hat{F}}\frac{\left(\hat{V}_N\left(\widehat{\pi\beta}\right)-2\beta C\hat{O}V_N\left(\widehat{\pi\beta},\hat{\pi}\right)+\beta^2\hat{V}_N(\hat{\pi})\right)}{\hat{V}_N(\hat{\pi})}$$

The result follows under Weak-IV asymptotics, by the continuous mapping theorem.

Limiting Distribution of $\hat{L}_{tF}$

By definition,

$$\hat{L}_{tF}=\frac{\sqrt{c_\alpha\left(\hat{F}\right)}}{\sqrt{q_{1-\alpha}}}\hat{L}_{IV}$$

The result follows under Weak-IV asymptotics, by the continuity of $c_\alpha(\cdot)$, and the continuous mapping theorem.

Limiting Distribution of $\hat{L}_{AR}$

We have shown above that the $(1-\alpha)$ AR confidence set is an interval if and only if $\hat{F}>q_{1-\alpha}$. If $\hat{F}<q_{1-\alpha}$, then the confidence set is the whole real line except for an interval of length $\hat{L}_{AR}$.

We have shown above that $\hat{L}_{AR}$ is related to $\hat{L}_{IV}$ by the relationship

$$\hat{L}_{AR}=\frac{\sqrt{1-\frac{q_{1-\alpha}}{\hat{F}}\left(1-\hat{\hat{\rho}}^2\right)}}{1-\frac{q_{1-\alpha}}{\hat{F}}}\hat{L}_{IV}$$

Note that, by definition

$$\hat{\hat{\rho}}=\frac{\hat{C}(Z\hat{u},Z\hat{v})}{\sqrt{\hat{V}(Z\hat{u})\hat{V}(Z\hat{v})}}$$

where $\hat{u}$ and $\hat{v}$ are the IV and first-stage residuals.

The following numerical relationship can be derived

$$\hat{t}_{AR}^2 = \frac{\hat{t}^2}{1+2\hat{\hat{\rho}}\frac{\hat{t}}{\sqrt{\hat{f}^2}}+\frac{\hat{t}^2}{\hat{f}^2}} = \frac{\hat{t}^2\hat{f}^2}{\hat{f}^2+2\hat{\hat{\rho}}\sqrt{\hat{F}}\hat{t}+\hat{t}^2}$$

Using this equation, we solve for $\hat{\hat{\rho}}$ and take its square, to obtain

$$\hat{\hat{\rho}}^2 = \frac{\left(\hat{t}^2\hat{f}^2-\hat{t}_{AR}^2\hat{t}^2-\hat{t}_{AR}^2\hat{f}^2\right)^2}{\left(2\hat{t}_{AR}^2\right)^2\hat{F}\hat{t}^2}$$

We can now substitute in the numerical relationship

$$\hat{t}^2 = \frac{\hat{t}_{AR}^2}{1-2\hat{\rho}\frac{\hat{t}_{AR}}{\hat{f}}+\frac{\hat{t}_{AR}^2}{\hat{f}^2}} = \frac{\hat{f}^2\hat{t}_{AR}^2}{\hat{f}^2-2\hat{\rho}\hat{f}\hat{t}_{AR}+\hat{t}_{AR}^2}$$

and with some simplification, one obtains

$$\hat{\hat{\rho}}^2 = \frac{\left(-\hat{t}_{AR}+\hat{\rho}\hat{f}\right)^2}{\left(\hat{f}^2-2\hat{\rho}\hat{t}_{AR}\hat{f}+\hat{t}_{AR}^2\right)}$$

which, under Weak-IV asymptotics and the continuous mapping theorem, converges in distribution to

$$\tilde{\rho}^2 = \frac{\left(-t_{AR}+\rho f\right)^2}{\left(f^2-2\rho t_{AR}f+t_{AR}^2\right)}$$

So $\hat{L}_{AR}$ converges in distribution to

$$L_{AR} = \frac{\sqrt{1-\frac{q_{1-\alpha}}{F}\left(1-\tilde{\rho}^2\right)}}{1-\frac{q_{1-\alpha}}{F}}L_{IV} = \frac{\sqrt{F}\sqrt{F-q_{1-\alpha}\left(1-\tilde{\rho}^2\right)}}{F-q_{1-\alpha}}L_{IV}$$

C.2 $E\left[L_{AR}|F>q_{1-\alpha}\right]=\infty$

Conditional Expected Length of *AR* interval. Let

$$\Omega = \mathbf{plim}N\begin{pmatrix} \hat{V}_N\left(\widehat{\pi\beta}\right) & C\hat{O}V\left(\hat{\pi},\widehat{\pi\beta}\right) \\ C\hat{O}V\left(\hat{\pi},\widehat{\pi\beta}\right) & \hat{V}_N\left(\hat{\pi}\right)\end{pmatrix},$$

the asymptotic variance-covariance matrix of the reduced form and first-stage coefficients, be positive definite. Then $E\left[L_{AR}|F>q\right]=\infty$

From above, we have

$$L_{AR} = \frac{\sqrt{F}\sqrt{F - q_{1-\alpha}(1-\tilde{\rho}^2)}}{F - q_{1-\alpha}} 2\sqrt{q_{1-\alpha}} \sqrt{1 - 2\rho\frac{t_{AR}}{f} + \frac{t_{AR}^2}{f^2}} \frac{1}{\sqrt{F}} \sqrt{V_\Omega}$$
$$= 2\sqrt{q_{1-\alpha}} \frac{\sqrt{f^2 - q_{1-\alpha}(1-\tilde{\rho}^2)}}{f^2 - q_{1-\alpha}} \sqrt{\frac{f^2 - 2\rho t_{AR} f + t_{AR}^2}{f^2}} \sqrt{V_\Omega}$$

with

$$\begin{pmatrix} t_{AR} \\ f \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ \frac{\pi}{\sqrt{AV(\hat{\pi})}} \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$
$$V_\Omega = \frac{(1,-\beta)'\,\Omega\,(1,-\beta)}{AV(\hat{\pi})}$$

We will show that

$$E\left[2\sqrt{q_{1-\alpha}}\frac{\sqrt{f^2-q_{1-\alpha}\left(1-\tilde{\rho}^2\right)}}{f^2-q_{1-\alpha}}\sqrt{\frac{f^2-2\rho t_{AR}f+t_{AR}^2}{f^2}}\sqrt{V_\Omega}|F>q_{1-\alpha}\right]= \quad (19)$$

$$\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{-\infty}^{\infty}\int_{\left(-\infty,-\sqrt{q_{1-\alpha}}\right)\cup\left(\sqrt{q_{1-\alpha}},\infty\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{x^2-q_{1-\alpha}\left(1-\tilde{\rho}\left(x,y\right)^2\right)}}{x^2-q_{1-\alpha}}$$
$$\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}\sqrt{V_\Omega}\phi_{f_0,\rho}\left(x,y\right)dxdy\geq$$

$$\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{x^2-q_{1-\alpha}\left(1-\tilde{\rho}\left(x,y\right)^2\right)}}{x^2-q_{1-\alpha}}$$
$$\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}\sqrt{V_\Omega}\phi_{f_0,\rho}\left(x,y\right)dxdy>$$

$$\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{x^2-q_{1-\alpha}\left(1-k_1\right)}}{x^2-q_{1-\alpha}}k_2k_3\sqrt{V_\Omega}dxdy\geq$$

$$\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{q_{1-\alpha}-q_{1-\alpha}\left(1-k_1\right)}}{x^2-q_{1-\alpha}}k_2k_3\sqrt{V_\Omega}dxdy=$$
$$\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{\sqrt{q_{1-\alpha}-q_{1-\alpha}\left(1-k_1\right)}}{\left(x-\sqrt{q_{1-\alpha}}\right)\left(x+\sqrt{q_{1-\alpha}}\right)}k_2k_3\sqrt{V_\Omega}dxdy\geq$$
$$\frac{1}{\Pr\left[F>q_{1-\alpha}\right]}\int_{\underline{y}}^{\underline{y}+\varepsilon}\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)}2\sqrt{q_{1-\alpha}}\frac{1}{\left(x-\sqrt{q_{1-\alpha}}\right)}\frac{\sqrt{q_{1-\alpha}k_1}}{2\sqrt{q_{1-\alpha}}+\varepsilon}k_2k_3\sqrt{V_\Omega}dxdy=\infty$$

where $\phi_{f_0,\rho}$ is the bivariate normal density with mean $(f_0,0)$, unit variances and correlation ρ, and $\tilde{\rho}\left(x,y\right)\equiv\frac{\left(-x+\rho y\right)^2}{\left(x^2-2\rho xy+y^2\right)}$, with both $\varepsilon>0$ and $\underline{y}$ chosen below.

In (19), the first equality (lines 1 and 2) holds by definition. The first inequality (lines 2 and 3) holds because the region of integration in the third line is a subset of the region for the second line. Deferring the second inequality momentarily, the third inequality (lines 4 and 5) holds because $\sqrt{x^2-q_{1-\alpha}\left(1-k_1\right)}>\sqrt{q_{1-\alpha}-q_{1-\alpha}\left(1-k_1\right)}$ because $x^2>q_{1-\alpha}$ in the region of integration. We expand a term in the denominator from lines 5 to 6 and the final inequality follows because $\frac{1}{x+\sqrt{q_{1-\alpha}}}\geq\frac{1}{2\sqrt{q_{1-\alpha}}+\varepsilon}$ when $x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)$. The final line holds because we will show it is equal to a positive constant multiplied by the integral

$\int_{\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{1}{\left(x-\sqrt{q_{1-\alpha}}\right)} dx$, which is infinite.

What remains is to show that the second inequality (lines 3 and 4) holds. Note first that $\sqrt{V_\Omega} > 0$ due to the positive definiteness of Ω. Furthermore, we will show that there always exists an integrating region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right) \times \left(\underline{y}, \underline{y}+\varepsilon\right)$ that lead to lower bounds $k_1, k_2, k_3 > 0$ for $\tilde{\rho}(x,y)^2$, $\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}$ and $\phi_{f_0,\rho}(x,y)$, respectively, on the region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right) \times \left(\underline{y}, \underline{y}+\varepsilon\right)$.

1. $\tilde{\rho}(x,y)^2 > k_1 > 0$. Consider the quantity

$$\tilde{\rho}(x,y)^2 = \frac{(-y+\rho x)^2}{(x^2-2\rho yx+y^2)}$$

We seek a region of x,y space that satisfies

$$\frac{(-y+\rho x)^2}{(x^2-2\rho yx+y^2)} \geq k_1 > 0$$

We restrict x to be in the interval $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right)$. We can keep the denomiator positive by restricting

$$y > \sup_{x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x+\sqrt{4\rho^2x^2-4}}{2}$$

In addition, we are seeking the values of y that satisfy

$$\begin{aligned}
(-y+\rho x)^2 - k_1\left(x^2-2\rho yx+y^2\right) &> 0 \\
y^2-2\rho yx+\rho^2x^2 - k_1\left(x^2-2\rho yx+y^2\right) &> 0 \\
y^2(1-k_1) - 2\rho yx(1-k_1) + x^2\left(\rho^2-k_1\right) &> 0
\end{aligned}$$

which is a quadratic inequality in y. We can choose $0 < k_1 < 1$ so that the function in the last line is convex in t_{AR}. So we can additionally restrict

$$y > \sup_{x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x(1-k_1)+\sqrt{4\rho^2x^2(1-k_1)^2-4(1-k_1)x^2(\rho^2-k_1)}}{2(1-k_1)}$$

So by setting

$$\underline{y} = \max\left(\sup_{x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x + \sqrt{4\rho^2x^2-4}}{2},\right.$$
$$\left.\sup_{x\in\left(\sqrt{q_{1-\alpha}},\sqrt{q_{1-\alpha}}+\varepsilon\right)} \frac{2\rho x\left(1-k_1\right) + \sqrt{4\rho^2x^2\left(1-k_1\right)^2 - 4\left(1-k_1\right)x^2\left(\rho^2-k_1\right)}}{2\left(1-k_1\right)}\right),$$

then for any $y > \underline{y}$, and $x \in \left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right)$ we have $\tilde{\rho}\left(x,y\right)^2 \geq k_1 > 0$, as desired.

2. $\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}} > k_2 > 0$. We established above that for $y > \underline{y}$ the numerator in the square root is positive. In the integrating region, the denominator is positive as well. Let k_2 be the infimum of $\sqrt{\frac{x^2-2\rho xy+y^2}{x^2}}$ over the region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right) \times \left(\underline{y}, \underline{y}+\varepsilon\right)$.

3. $\phi_{f_0,\rho}\left(x,y\right) > k_3 > 0$. The bivariate density is strictly positive. Let k_3 be the infimum of $\phi_{f_0,\rho}\left(x,y\right)$ over the region $\left(\sqrt{q_{1-\alpha}}, \sqrt{q_{1-\alpha}}+\varepsilon\right) \times \left(\underline{y}, \underline{y}+\varepsilon\right)$

⊠

C.3 $E\left[L_{tF}|F > q_{1-\alpha}\right] < \infty$

Conditional Expected Length of tF interval: $E\left[L_{tF}|F > q_{1-\alpha}\right] < \infty$**.**

As shown above

$$L_{tF} \equiv \frac{\sqrt{c_\alpha\left(F\right)}}{\sqrt{q_{1-\alpha}}} 2\sqrt{q_{1-\alpha}}\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_\Omega}$$
$$= \sqrt{c_\alpha\left(F\right)}2\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_\Omega}$$

The conditional expectation of interest is

$$E\left[2\sqrt{c_\alpha\left(F\right)}\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_\Omega}|F > q_{1-\alpha}\right]$$

We start by considering the conditional expectation

$$E\left[2\sqrt{c_{\alpha}(F)}\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_{\Omega}}|F=F'\right]$$

Since $c_{\alpha}(F)$ only depends on F, this is equivalent to

$$2\sqrt{c_{\alpha}(F')}E\left[\sqrt{1-2\rho\frac{t_{AR}}{f}+\frac{t_{AR}^2}{f^2}}\frac{1}{\sqrt{F}}\sqrt{V_{\Omega}}|F=F'\right]$$

Let us consider the expectation conditional on F

$$E\left[\sqrt{\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^2}{F^2}}\sqrt{V_{\Omega}}|F=F'\right]$$

Consider the conditional expectation of the random variable inside the square root:

$$E\left[\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^2}{F^2}|F=F'\right]$$

which can be expressed as

$$\frac{1}{(F')^2}\left[F'-2\rho\sqrt{F'}E\left[t_{AR}|F=F'\right]+E\left[t_{AR}^2|F=F'\right]\right]$$

Consider that

$$\begin{aligned}
E\left[t_{AR}|F=F'\right] &= \left(\rho\left(-\sqrt{F'}-f_0\right)\right)\phi\left(-\sqrt{F'}\right)+\left(\rho\left(\sqrt{F'}-f_0\right)\right)\phi\left(\sqrt{F'}\right)\\
E\left[t_{AR}^2|F=F'\right] &= \left(1-\rho^2+\left(\rho\left(-\sqrt{F'}-f_0\right)\right)^2\right)\phi\left(-\sqrt{F'}\right)\\
&\quad+\left(1-\rho^2+\left(\left(\rho\left(\sqrt{F'}-f_0\right)\right)\right)^2\right)\phi\left(\sqrt{F'}\right)
\end{aligned}$$

Since each of these expressions is bounded on $F'>q_{1-\alpha}$, $E\left[\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^2}{F^2}|F=F'\right]$ is thus bounded on $F'>q_{1-\alpha}$ by some constant $\bar{F}$. Due to Jensen's inequality, we obtain

$$2\sqrt{c_\alpha(F')}E\left[\sqrt{\frac{1}{F}-\frac{2\rho t_{AR}}{fF}+\frac{t_{AR}^2}{F^2}}\sqrt{V_\Omega}|F=F'\right]\leq 2\sqrt{c_\alpha(F')}\bar{F}$$

Therefore, for $F'>q_{1-\alpha}$, the function $E[L_{tF}|F=F']$ is bounded above by the function $2\sqrt{c_\alpha(F')}\bar{F}$. Therefore,

$$\begin{aligned}E[L_{tF}|F>q_{1-\alpha}] &= \frac{1}{\Pr[F>q_{1-\alpha}]}\int_{q_{1-\alpha}}^{\infty}E\left[L_{tF}|F=F'\right]\omega\left(F'\right)dF' \\ &\leq \frac{1}{\Pr[F>q_{1-\alpha}]}\int_{q_{1-\alpha}}^{\infty}2\sqrt{c_\alpha(F')}\bar{F}\omega\left(F'\right)dF'\end{aligned}$$

where ω is the density of F.

Finally, from the proof of Lemma 9 we know that $c_\alpha(F)(F-q_{1-\alpha})$ is continuous on $(q_{1-\alpha},q_{1-\alpha}+\varepsilon]$. This function can be extended continuously at $F=q_{1-\alpha}$ by the asymptotic approximation of Lemma 9. As this function is defined on the compact set $[q_{1-\alpha},q_{1-\alpha}+\varepsilon]$, it is uniformly continuous on this set.

and hence bounded above by some finite value M on $[q_{1-\alpha},q_{1-\alpha}+\varepsilon]$. The density ω is also bounded above by K in the same interval. Therefore

$$\int_{q_{1-\alpha}}^{q_{1-\alpha}+\varepsilon}\sqrt{c_\alpha(F')}\omega\left(F'\right)dF'\leq\int_{q_{1-\alpha}}^{q_{1-\alpha}+\varepsilon}\frac{\sqrt{M}}{\sqrt{F-q_{1-\alpha}}}KdF'<\infty$$

Since $\sqrt{c_\alpha(F')}$ is bounded on $(q_{1-\alpha}+\varepsilon,\infty)$ by say M', then $\int_{q_{1-\alpha}+\varepsilon}^{\infty}\sqrt{c_\alpha(F')}\omega(F')dF'<M'\int_{q_{1-\alpha}+\varepsilon}^{\infty}\omega(F')dF'\leq M'<\infty$, which completes the proof. ⊠