ORF 523	PROBLEM SET 3	Spring 2025, Princeton University
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Due on March 4, 2025, at 1:30pm ET, on Gradescope		

Problem 1: Norms defined by convex sets

Define M_C as the following function of a convex set C in $\mathbb{R}^n$:

$$M_C(x) = \inf\{t > 0 \mid \frac{x}{t} \in C\},$$

over the domain

$$dom(M_C) = \{x \in \mathbb{R}^n \mid \frac{x}{t} \in C \text{ for some } t > 0\}.$$

1. Show that M_C is a convex function.
2. Suppose C is also compact, origin symmetric ($x \in C$ if and only if $-x \in C$), and has nonempty interior. Show that M_C is a norm over $\mathbb{R}^n$. What is its unit ball?
3. Show that an even degree homogeneous polynomial is convex if and only if it is quasiconvex. (Hint: use what you proved in the previous parts of this question.)

Problem 2: Totally unimodular matrices

Let A be an $m \times n$ integral matrix. Show that A is totally unimodular if and only if for every integral vector b, the polyhedron $\{x \mid x \geq 0, Ax \leq b\}$ is integral. (Hint: if A is not totally unimodular, use the inverse of a submatrix which does not have determinant $\{0, -1, +1\}$ to construct an integer vector b that generates a non-integral vertex in the polyhedron.)

Problem 3: Doubly stochastic matrices

A real $n \times n$ matrix Q is said to be *doubly stochastic* if its entries are nonnegative and its rows and columns all sum up to 1. We say that Q is a *permutation* matrix if it has exactly one 1 in every row and every column and zeros everywhere else. Show that every doubly stochastic matrix is a convex combination of permutation matrices. (Hint: you can use the fact that any point in a bounded polyhedron is a convex combination of its vertices.)