ORF 523	Problem set 2	Spring 2025, Princeton University
Instructor: A.A. Ahmadi		
AIs: Y. Hua, J.J. Macosko		
Due on February 25, 2025, at 1:30pm ET, on Gradescope		

For all problems that involve coding, please include your code.

Problem 1: True or False?

Specify whether each of the following statements is true or false and provide either a proof or a counterexample depending on your answer. Let S be a set in $\mathbb{R}^n$.

1. The convex hull of S is the intersection of all convex sets that contain S.

2. If S is closed, then the convex hull of S is closed.

3. If S is bounded, then the convex hull of S is bounded.

4. If S is compact, then the convex hull of S is compact.
(You may want to use the following fact from analysis: the image of a compact set under a continuous mapping is compact.)

5. The sum of two quasiconvex functions is quasiconvex.

6. A quadratic function $f(x) = x^TQx + b^Tx + c$ is convex if and only if it is quasiconvex.
(You can use the fact that f is convex if and only if $Q \succeq 0$ if you need to.)

7. Any closed convex set $\Omega \subseteq \mathbb{R}^n$ can be written as $\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ for some convex function $g : \mathbb{R}^n \to \mathbb{R}$.

8. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex on a convex set $S \subseteq \mathbb{R}^n$, then f is continuous on S.

9. Every local minimum of a quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$ is a global minimum.

10. Every strict local minimum of a quasiconvex function $f : \mathbb{R}^n \to \mathbb{R}$ is a strict global minimum.

Problem 2: CVX(PY) warmup / Minimum fuel optimal control

(Boyd&Vandenberghe, Problem 4.16)

We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1,$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e. $x(0) = 0$. The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final or target state. The function $f : \mathbb{R} \to \mathbb{R}$ is the *fuel use map* for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \le 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

(a) Formulate the minimum fuel optimal control problem as a linear program, i.e., a convex optimization problem with affine objective and constraint functions.

(b) Solve the minimum fuel optimal control problem using CVX or CVXPY for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \qquad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad N = 30.$$

Plot the actuator signal $u(t)$ as a function of time t using the function `stairs` (In Python you need to import `matplotlib` first). You are allowed to let CVX or CVXPY formulate the LP for you, but it's a good idea to check the answer against the LP that you formulated in the previous part.