ORF 363/COS 323	PROBLEM SET 7	Fall 2024, Princeton University
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Due on Tuesday, November 26, 2023, at 1:30pm EST, on Gradescope		

For all problems that involve coding, please include your code.

Problem 1: Setting the odds in your favor with semidefinite programming

As the CEO of TigerCasino in Vegas, you are introducing a new game on your floor. In this game, a player rolls a die twice and receives q_{ij} dollars from the casino if the die shows i on one roll and j on the other (the order does not matter). The matrix $Q = (q_{ij})_{1\leq i,j\leq 6}$ is announced to the player:

$$Q = 100 \times \begin{pmatrix} 4 & -2 & 1 & -1 & -2 & 1 \\ -2 & 4 & 1 & -2 & -2 & -1 \\ 1 & 1 & 4 & -2 & 1 & -1 \\ -1 & -2 & -2 & 4 & 1 & -1 \\ -2 & -2 & 1 & 1 & 4 & -1 \\ 1 & -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

Gamblers are leaving the Bellagios and rushing to your table because if the die was fair (i.e., had a probability of $\frac{1}{6}$ assigned to each outcome of a roll), they would make \$11.11 in expectation in every play. Little do they know, however, that you have been using your optimization knowledge to optimally bias the die and maximize the profit of TigerCasino. Let x_i be the probability that the die comes out i. The optimization problem of interest to you is:

$$\begin{aligned} \min_{x\in\mathbb{R}^6} \quad & x^T Q x \\ \text{s.t.} \quad & x \geq 0 \\ & \sum_{i=1}^{6} x_i = 1. \end{aligned} \tag{1}$$

The constraints make sure that x is a valid probability vector and the objective function is the expected payoff of the player in every play.

(a) Is problem (1) a convex optimization problem? Why or why not?

(b) Recall that a matrix $A \in \mathbb{S}^{n\times n}$ is copositive if $x^T Ax \geq 0$ for all $x \geq 0$. Denote the set of $n \times n$ copositive matrices by $\mathcal{C}_n$. Show that the optimal value of (1) is equal to the optimal value of the following problem:

$$\begin{aligned} \max_{t\in\mathbb{R}} \quad & t \\ \text{s.t.} \quad & Q - tJ \in \mathcal{C}_6. \end{aligned} \tag{2}$$

Here, $J \in \mathbb{S}^{6\times 6}$ is the all-ones matrix.

(c) Denote the optimal value of (1) (or equivalently (2)) by OPT. Denote the optimal value of the semidefinite program

$$\begin{aligned} \max_{t\in\mathbb{R}, N\in\mathbb{S}^{6\times 6}} \quad & t \\ \text{s.t.} \quad & Q - tJ - N \succeq 0 \\ & N \geq 0 \end{aligned} \tag{3}$$

by SDP_{OPT}. (Here, "$\geq$" denotes an entrywise nonnegativity constraint and "$\succeq$" denotes a positive semidefiniteness constraint.) Show that $SDP_{OPT} \leq OPT$.

(d) Report SDP_{OPT} by solving (3) in CVX or CVXPY. Show that $SDP_{OPT} = OPT$ by presenting a vector $x^* \in \mathbb{R}^6$ that is feasible to (1) and makes the objective function of (1) equal to SDP_{OPT}. (Hint: you may wish to start with an eigenvector associated with the smallest eigenvalue of $Q - t^*J - N^*$, where (t^*, N^*) form an optimal solution to (3).) What probability does your optimal die assign to each of its six outcomes? What is the expected win/loss of TigerCasino in dollars every time a player plays this game?

Problem 2: Minimum fuel optimal control. (Boyd&Vandenberghe)
We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1,$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e. $x(0) = 0$. The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final or target state. The function $f : \mathbb{R} \to \mathbb{R}$ is the *fuel use map* for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

(a) Formulate the minimum fuel optimal control problem as a linear program. (Your linear program does not need to be written in standard form.)

(b) Solve the minimum fuel optimal control problem using CVX for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \qquad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad N = 30.$$

Plot the actuator signal $u(t)$ as a function of time t.

Problem 3: Nearest correlation matrix

You are the Head of Quantitative Strategies at HoneyMoney Technologies LLC, a new hedge fund firm in NYC whose proprietary optimization algorithms has Wall Street raving. Your main competitor, RenaissancE Technologies[1], has sent in a spy, disguised as a summer intern, to interfere with your investments. The spy has gotten his hands on your *correlation matrix* C of n important stocks[2], to which he has added some random noise, leaving you with a matrix $\hat{C}$. We remark that to be a valid correlation matrix, a matrix must be symmetric, positive semidefinite, and have all diagonal entries equal to one. The spy has been careful enough to make sure that the resulting matrix $\hat{C}$ is symmetric and has ones on the diagonal, but he hasn't noticed that his change has made $\hat{C}$ not positive semidefinite.

(a) Suppose we have

$$\hat{C} = \begin{pmatrix} 1.00 & -0.76 & 0.07 & -0.96 \\ -0.76 & 1.00 & 0.18 & 0.07 \\ 0.07 & 0.18 & 1.00 & 0.41 \\ -0.96 & 0.07 & 0.41 & 1.00 \end{pmatrix}.$$

Using CVX or CVXPY, recover the original matrix by finding the nearest correlation matrix to $\hat{C}$ in Frobenius norm (i.e., the correlation matrix C that minimizes $\|C - \hat{C}\|_F$). Give your optimal solution.

(b) Show that for any symmetric matrix $\hat{C}$, the problem of finding the closest correlation matrix to $\hat{C}$ in Frobenius norm has a unique solution. (You can use the fact that an optimal solution to this problem exists without proof.)

[1] Not to be confused with Renaissance Technologies that would never do such a thing.

[2] If you are curious, the correlation matrix is an $n \times n$ symmetric matrix used frequently in investment banking. Its (i, j)-th entry is a number between -1 and 1, with numbers close to 1 meaning that stocks i and j are likely to move up together, close to -1 meaning that the two stocks are likely to move in opposite directions, and close to zero meaning that they are likely uncorrelated. The problem of finding the closest correlation matrix to a given matrix is an important problem in financial engineering; see e.g. this article.